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AN INFINITE FAMILY OF SYMMETRIC DESIGNS

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In this paper, using the construction method of [3], we show that if $q>2$ is a prime power such that there exists an affine plane of order $q-1$, then there exists a strongly divisible $2-((q-1)(q^h-1), q^{h-1}(q-1), q^{h-1})$ design for every $h \geq 2$. We show that these quasi-residual designs are embeddable, and hence establish the existence of an infinite family of symmetric $2-(q^{h+1}-q+1, q^h, q^{h-1})$ designs. This construction may be regarded as a generalisation of the construction of [1, Chapter 4, Section 1] and [4].

1. Introduction

The study of symmetric 2-designs, or "Symmetrical BIBD's", can be divided into two main sections. Firstly, the proving of theorems showing the non-existence of symmetric designs with given parameters, see, for instance [5], [7]; and, secondly, the construction of designs for certain parameter sets. Some examples of infinite families of symmetric 2-designs are: finite projective planes (a finite projective plane may be regarded as a symmetric design with $\lambda = 1$); Hadamard Designs (symmetric 2-$(4\lambda+3, 2\lambda+1, \lambda)$ designs) and symmetric designs satisfying the condition $v = 4(k-\lambda)$ obtained from the construction method of [8]. A partial list of known results on symmetric 2-designs may be found in [6].

In this paper we use a well-known family of group divisible 1-designs in conjunction with affine 2-designs of appropriate parameters, to construct a family of strongly divisible 1-designs, using the method of [3]. These strongly divisible designs are quasi-residual 2-designs in the case when the affine 2-designs are affine planes, and, in this case, we show that they are the residual designs of a family of symmetric 2-designs. Infinitely many of these designs have parameters for which no design was previously known to exist.

2. The construction

For definitions and results used, see [2], [3] and [6]. Suppose $q>2$ is a prime power, and let $h>1$ be any integer. Then put $\mathbf{A}' = \mathbf{A}_{h-1}(h, q)$, the design consisting of the points and hyperplanes of h-dimensional affine geometry over

$GF(q)$; (see, for instance, [6]). Choose some point P of $\mathbf{A}'$ and, using the notation of [6], let $\mathbf{S}=(\mathbf{A}')^P$.

Then it is not difficult to show that $\mathbf{S}$ is a 1-$(q^h-1, q^{h-1}, q^{h-1})$ design, admitting a strong tactical division $T(\mathbf{S})$ with n point and block classes ($n=(q^h-1)/(q-1)$) of $q-1$ points and blocks each. The point classes of $T(\mathbf{S})$ are the lines of $\mathbf{A}'$ which contain P, (with P removed in each case), and the block classes are just the parallel classes of $\mathbf{A}'$ with the block containing P omitted.

$\mathbf{S}$ has connection and intersection numbers λ'_{ij}, ρ'_{ij} where $\rho'=\lambda'=\rho'_{ii}=\lambda'_{ii}=0$ ($1\leq i\leq n$), and $\rho'_{ij}=\lambda'_{ij}=q^{h-2}$ ($1\leq i, j\leq n$, $i\neq j$). Finally it is clear that no block of $\mathbf{S}$ contains all the points of a point class of $\mathbf{S}$. (Note that to construct $\mathbf{S}$ with the above properties, we needed only that $\mathbf{A}'$ was an affine design with constant line size, and that $\mathbf{A}'$ was smooth. Hence we could replace $\mathbf{A}_1(2,q)$ by an arbitrary affine plane of order q.)

We now require the following from [3].

Result 1. Suppose there exists

(i) A 2-$(\mu m^2, \mu m, (\mu m-1)/(m-1))$ affine design $\mathbf{A}$; and

(ii) A 1-(mn, k', k') design $\mathbf{S}$ admitting a strong tactical division with n point and block classes of m points and blocks each, with connection and intersection numbers $\lambda'=\lambda'_{ii}$, λ'_{ij}, $\rho'=\rho'_{ii}$, ρ'_{ij}; and such that no block of $\mathbf{S}$ contains all the points from a point class of the strong tactical division.

Then there exists a 1-$(\mu m^2 n, \mu m k', (\mu m^2-1)k'/(m-1))$ design $\mathbf{D}$ admitting a strong tactical division $T(\mathbf{D})$ with n point classes of μm^2 points each, and $(\mu m^2-1)n/(m-1)$ block classes of m blocks each. The classes admit a labelling such that the connection and intersection numbers are

$$\lambda=\lambda_{ii}=(\mu m-1)k'/(m-1)+\mu m\lambda'; \qquad \lambda_{ij}=(\mu m^2-1)\lambda'_{ij}/(m-1);$$

$\rho=\rho_{ii}=\mu m\rho'$ and ρ_{ij}, where

$$\rho_{ij}=\mu m\rho'_{tu}, \quad i\equiv t\not\equiv u\equiv j \pmod c),\ 1\leq t, u\leq c;$$

and

$$\rho_{ij}=(\rho'+k'/m)\mu m, \quad i\equiv j \pmod c),\ i\neq j.$$

Theorem 1. *If there exists an affine*

$$2\text{-}(\mu(q-1)^2, \mu(q-1), (\mu(q-1)-1)/(q-2)) \quad \textit{design } \mathbf{A},$$

and $q>2$ is a prime power, then for every $h\geq 2$ there exists a

$$1\text{-}(\mu(q-1)(q^h-1), \mu q^{h-1}(q-1),\ q^{h-1}(\mu(q-1)^2-1)/(q-2)) \quad \textit{design } \mathbf{D}$$

admitting a strong tactical divison $T(\mathbf{D})$ with n point classes each of $\mu(q-1)^2$ points and c block classes ($c=(\mu(q-1)^2-1)(q^h-1)/(q-1)(q-2)$) each of $q-1$ blocks. The classes of $T(\mathbf{D})$ may be labelled so that the connection and intersection numbers *are*

$$\lambda = \lambda_{ii} = (\mu(q-1)-1)q^{h-1}/(q-2), \quad (1 \leq i \leq n);$$
$$\lambda_{ij} = (\mu(q-1)^2-1)q^{h-2}/(q-2), \quad (1 \leq i, j \leq n, i \neq j);$$

$\rho = \rho_{ii} = 0, (1 \leq i \leq c)$; *and* ρ_{ij}, *where*

$$\rho_{ij} = \mu q^{h-2}(q-1), \quad i \not\equiv j \pmod n, \ (1 \leq i, j \leq c);$$

and

$$\rho_{ij} = \mu q^{h-1}, \quad i \equiv j (\bmod n), (1 \leq i, j \leq c, i \neq j).$$

Proof. Since $q > 2$ is a prime power let $\boldsymbol{S}$ be as above. Then $\boldsymbol{S}$ and $\mathbf{A}$ satisfy the conditions of Result 1 and the Theorem follows. □

Corollary. *If there exists an affine plane of order* $q-1$, *and* $q > 2$ *is a prime power, then for every* $h \geq 2$, *there exists a*

$$2\text{-}((q-1)(q^h-1), q^{h-1}(q-1), q^{h-1}) \quad \textit{design } \mathbf{D},$$

admitting a strong tactical decomposition $T(\mathbf{D})$ *with n point classes each of* $(q-1)^2$ *points, and c block classes* $(c = q(q^h-1)/(q-1))$ *each of* $q-1$ *blocks. The intersection numbers are* $\rho = \rho_{ii} = 0$ $(1 \leq i \leq c)$; *and* ρ_{ij}, *where*

$$\rho_{ij} = q^{h-2}(q-1), \quad i \not\equiv j \pmod n, \quad (1 \leq i, j \leq c);$$

and

$$\rho_{ij} = q^{h-1}, \quad i \equiv j \pmod n, \quad (1 \leq i, j \leq c, i \neq j).$$

Proof. $\boldsymbol{D}$ of Theorem 1 is a 2-design if and only if

$$(\mu(q-1)^2-1)q^{h-2}/(q-2) = (\mu(q-1)-1)q^{h-1}/(q-2),$$

i.e. $\mu = 1$ or $q = 1$; i.e. $\boldsymbol{D}$ is a 2-design if and only if $\mathbf{A}$ is an affine plane (since $q > 2$). □

3. The embedding

We first require

Result 2 ([2, Corollary 6.3]). A quasi-residual

$$2\text{-}((k-1)(k-\lambda)/\lambda, k-\lambda, \lambda) \quad \text{design } \boldsymbol{D}$$

with three intersection numbers: 0, $\lambda(k-\lambda)/k$ and k/m is embeddable in a symmetric 2-(v, k, λ) design if and only if there exists a strongly resolvable 2-$(k, \lambda, (\lambda-1)/m)$ design $\bar{\boldsymbol{D}}$.

We may now state

Theorem 2. *If there exists an affine plane of order* $q-1$, *and* $q>2$ *is a prime power, then for every* $h\geqslant 2$, *there exists a* $2\text{-}(q^{h+1}-q+1, q^h, q^{h-1})$ *design.*

Proof. By the Corollary of Theorem 1, for every $h\geqslant 2$ there exists a

$$2\text{-}((q-1)(q^h-1), q^{h-1}(q-1), q^{h-1}) \quad \text{design } \boldsymbol{D}$$

with intersection numbers: 0, $q^{h-2}(q-1)$ and q^{h-1}. Hence, by Result 2, $\boldsymbol{D}$ is embeddable in a symmetric $2\text{-}(q^{h+1}-q+1, q^h, q^{h-1})$ design if and only if there exists a strongly resolvable

$$2\text{-}(q^h, q^{h-1}, (q^{h-1}-1)/(q-1)) \quad \text{design } \bar{\boldsymbol{D}}.$$

But such a design always exists, namely $\bar{\boldsymbol{D}}=\boldsymbol{A}_{h-1}(h,q)$, and the theorem follows. □

Remark. Hence, since there exists an affine plane of order $q-1$ whenever $q-1$ is a prime power, we have shown that whenever q and $q-1$ are prime powers, there exists an infinite family of symmetric 2-designs with the above parameters, since h may be chosen arbitrarily.

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