CHAPTER 2 - POINT DIVISIONS

In this chapter we generalise the concept of group divisibility by introducing the notion of a point division of a 1-structure. We redefine group divisible designs in terms of point divisions, and useful results are obtained for group divisible designs using the more general notion; (see, in particular, Chapter 3 below).

In the first section point divisions are defined and some basic results are obtained. Section 2.2 mainly consists of generalisations of Bose and Connor's results for GD designs to structures admitting point divisions; the corresponding results for GD designs being obtained as corollaries to the more general results.

In section 2.3 we generalise SRGD designs to Semi-regular Point Divisible (SRPD) designs and analagous results are obtained; necessary and sufficient conditions are also given for a SRPD design to be SRGD.

The fourth section indicates the links between point divisions and tactical decompositions, and, finally, in 2.5 we consider 2-designs whose duals admit point divisions, deriving some necessary and sufficient conditions for a design to be a strongly resolvable 2-design.

Throughout this chapter we will assume that $\underline{S}$ is a 1-(v,k,r) structure and $\underline{D}$ is a 1-(v,k,r) design, unless otherwise stated.

2.1 Definitions and Basic Results

A Point Division (or just P Division) of $\underline{S}$, is a partition $\underline{P}_1,\ldots,\underline{P}_d$ of the points of $\underline{S}$, such that the connection number of two distinct points from classes $\underline{P}_i$ and $\underline{P}_j$ depends only on the choice of i and j and is denoted by λ_{ij}. We will write ℓ_i for $|\underline{P}_i|$. Then we have the following immediate generalisation of Result 1.4.1(i).

Lemma 2.1.1 If $\underline{P}_1,\ldots,\underline{P}_d$ is a point division of $\underline{S}$, then

$$v = \sum_{i=1}^{d} \ell_i .$$

A point division is said to be trivial if $v = d$ (i.e. if every point class contains just one point); every 1-structure admits the trivial point division. A Constant Lambda Point Division (or a CLP Division) of $\underline{S}$ is a non-trivial point division with a constant λ such that $\lambda_{ii} = \lambda$ for every i with $\ell_i > 1$.

As previously, these definitions are by no means standard. Adhikary, in [1], calls a structure admitting a non-trivial P Division, "Generalised Group Divisible", and Beker (see, for instance, [8]) introduced the term point division originally to describe what we call here a CLP Division.

Using our terminology, a Tactical Division of $\underline{S}$ is a tactical decomposition whose point classes form a CLP Division. A Group Division of $\underline{S}$ is a CLP Division such that there exists some constant λ' $(\lambda' \neq \lambda)$ with $\lambda_{ij} = \lambda'$ for every i,j $(i \neq j)$; and then a GD design is just a 1-design which admits a group division.

Since any refinement of the partition $\underline{P}_1,\ldots,\underline{P}_d$ of a P Division also forms a P Division we now introduce the concept of maximality. If $\underline{P}_1,\ldots,\underline{P}_d$ is a P Division of $\underline{S}$, then it is said to be <u>maximal</u> if the only P Division of $\underline{S}$ having $\underline{P}_1,\ldots,\underline{P}_d$ as a refinement is $\underline{P}_1,\ldots,\underline{P}_d$ itself. Then immediately we have :

<u>Lemma 2.1.2</u> A group division is a maximal point division.

We now show :

<u>Lemma 2.1.3</u> Every 1-structure $\underline{S}$ has a unique maximal P Division; (apart from relabelling of the point classes).

<u>Proof</u> Clearly every 1-structure admits the trivial P Division and so admits a maximal P Division.

Suppose $\underline{P}_1,\ldots,\underline{P}_d$ and $\underline{Q}_1,\ldots,\underline{Q}_{d'}$ are maximal point divisions of $\underline{S}$. Then let $\underline{R}_1,\ldots,\underline{R}_e$ be the partition of the points of $\underline{S}$ formed by taking all the non-empty intersections of the form $\underline{P}_i \cap \underline{Q}_j$; $(1 \leq i \leq d,\ 1 \leq j \leq d')$. $\underline{R}_1,\ldots,\underline{R}_e$ is then a refinement of both $\underline{P}_1,\ldots,\underline{P}_d$ and $\underline{Q}_1,\ldots,\underline{Q}_{d'}$ and hence is a P Division of $\underline{S}$.

Suppose $\underline{R}_1,\ldots,\underline{R}_e$ is not maximal. Then there exists a P Division $\underline{S}_1,\ldots,\underline{S}_f$ of $\underline{S}$ having $\underline{R}_1,\ldots,\underline{R}_e$ as a refinement, and hence the point classes $\underline{S}_i$ consist of unions of classes $\underline{R}_j$, and $f<e$. Every refinement of $\underline{S}_1,\ldots,\underline{S}_f$ is a P Division and so (without loss of generality) we assume that $\underline{R}_1 \cup \underline{R}_2,\ \underline{R}_3,\ \underline{R}_4,\ldots,\underline{R}_e$ is a refinement of $\underline{S}_1,\ldots,\underline{S}_f$ and hence is a P Division of $\underline{S}$. In addition let $\underline{R}_1 = \underline{P}_i \cap \underline{Q}_j$ and $\underline{R}_2 = \underline{P}_{i'} \cap \underline{Q}_{j'}$

Assume $i \neq i'$ and suppose $\underline{P}_1,\ldots,\underline{P}_d$ has connection numbers λ_{ij}, $(1 \leq i,j \leq d)$. Pick $X \in \underline{R}_1$ and $Y \in \underline{R}_2$ and consider any $Z \in \underline{P}_\ell$ for some $\ell \neq i,i'$. Since $\underline{R}_1 \subseteq \underline{P}_i$ and $\underline{R}_2 \subseteq \underline{P}_{i'}$, the connection numbers of X,Z and Y,Z are $\lambda_{i\ell}$ and $\lambda_{i'\ell}$ respectively. But $X,Y \in \underline{R}_1 \cup \underline{R}_2$ which forms a class of a P Division of $\underline{S}$, and so these connection numbers must be the same, i.e. $\lambda_{i\ell} = \lambda_{i'\ell}$ for every $\ell \neq i,i'$.

Now suppose $|\underline{P}_i| > 1$, i.e. suppose λ_{ii} is defined. Pick $W \in \underline{P}_i$, $W \neq X$. Then the connection numbers of X,W and Y,W are λ_{ii} and $\lambda_{ii'}$ respectively. But since $\underline{R}_1 \cup \underline{R}_2$ is a class of a P Division the connection numbers must be equal, and so $\lambda_{ii} = \lambda_{ii'}$. Similarly if $|\underline{P}_{i'}| > 1$ we see that $\lambda_{i'i'} = \lambda_{ii'}$.

So the partition of the points of $\underline{S}$ obtained from $\underline{P}_1,\ldots,\underline{P}_d$ by taking the union of $\underline{P}_i$ and $\underline{P}_{i'}$ as one class, and leaving the other classes unchanged forms a P Division, contradicting the assumption that $\underline{P}_1,\ldots,\underline{P}_d$ is maximal.

Similarly, if we assume that $j \neq j'$, we can deduce that $\underline{Q}_1,\ldots,\underline{Q}_{d'}$ is not maximal and again obtain a contradiction.

So we have shown that $\underline{R}_1,\ldots,\underline{R}_e$ is a maximal P Division of $\underline{S}$, and since it is a refinement of both $\underline{P}_1,\ldots,\underline{P}_d$ and $\underline{Q}_1,\ldots,\underline{Q}_{d'}$, we then have $\{\underline{P}_1,\ldots,\underline{P}_d\} = \{\underline{R}_1,\ldots,\underline{R}_e\} = \{\underline{Q}_1,\ldots,\underline{Q}_{d'}\}$. ¤

Remark From Lemma 2.1.2 we see that a group division of a 1-structure is unique. As stated above P Divisions are clearly not unique, (unless the only P Division admitted is the trivial one), but below we show that for a certain class of P Divisions, a similar result to Lemma 2.1.2 does hold.

We now give some elementary combinatorial identities for P Divisions.

Lemma 2.1.4 Suppose $\underline{S}$ admits a P Division $\underline{P}_1,\ldots,\underline{P}_d$, and let $x_1,\ldots,x_b$ be an arbitrary labelling of the blocks of $\underline{S}$. If $s^i_{uw} = |x_u \cap x_w \cap \underline{P}_i|$ then for every i,j ($1 \leq i, j \leq d$) :-

(i) $\displaystyle\sum_{u=1}^{b} s^i_{uu} = r\,\ell_i$;

(ii) $\sum_{u=1}^{b} s^i_{uu}\, s^j_{uu} = \ell_i \ell_j \lambda_{ij} + \ell_i (r-\lambda_{ii})\delta_{ij}$;

(iii) $\sum_{u=1}^{b} \sum_{w=1}^{b} s^i_{uw} = r^2 \ell_i$;

and (iv) $\sum_{u=1}^{b} \sum_{w=1}^{b} s^i_{uu}\, s^j_{ww} = r^2 \ell_i \ell_j$.

<u>Proof</u> Let A be an incidence matrix for $\underline{S}$ associated with the point division $\underline{P}_1, \ldots, \underline{P}_d$, i.e.

$$A = \begin{pmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_d \end{pmatrix}$$

where A_i is the $\ell_i \times b$ incidence matrix whose rows correspond to the points of $\underline{P}_i$ and whose columns correspond to the blocks of $\underline{S}$.

(i) Consider the matrix identity :-

$(\underline{j}A_i)\underline{j}^T = \underline{j}(A_i\underline{j}^T)$.

$\underline{j}A_i = (s^i_{11}\ s^i_{22} \cdots s^i_{bb})$.

So $(\underline{j}A_i)\underline{j}^T = \sum_{u=1}^{b} s^i_{uu}$.

$A_i\underline{j}^T = r\underline{j}^T$, and so $\underline{j}(A_i\underline{j}^T) = r\underline{j}\underline{j}^T = r\ell_i$, and (i) follows.

(ii) Consider the matrix identity :-

$(\underline{j}A_i)(\underline{j}A_j)^T = \underline{j}(A_iA_j^T)\underline{j}^T$.

By above, $(\underline{j}A_i)(\underline{j}A_j)^T = \sum_{u=1}^{b} s^i_{uu}\, s^j_{uu}$.

$$A_iA_j^T = \begin{cases} \lambda_{ij}\, J & \text{if } i \neq j \\ \lambda_{ii}\, J + (r-\lambda_{ii})I & \text{if } i = j \end{cases} \quad \text{, and so}$$

$$\underline{j}(A_iA_j)^T\underline{j}^T = \begin{cases} \ell_i \ell_j \lambda_{ij} & \text{if } i \neq j \\ \ell_i^{\,2}\, \lambda_{ii} + \ell_i(r-\lambda_{ii}) & \text{if } i = j \end{cases} \text{, and (ii) follows.}$$

(iii) Consider the matrix identity :

$$\underline{j}(A_i^TA_i)\underline{j}^T = (\underline{j}A_i^T)(\underline{j}A_i^T)^T .$$

$$A_i^TA_i = \begin{pmatrix} s^i_{11} & s^i_{12} & \cdots & s^i_{1b} \\ s^i_{21} & s^i_{22} & \cdots & s^i_{2b} \\ \vdots & \vdots & & \vdots \\ s^i_{b1} & s^i_{b2} & \cdots & s^i_{bb} \end{pmatrix} .$$

So $\underline{j}(A_i^TA_i)\underline{j}^T = \sum_{u=1}^{b} \sum_{w=1}^{b} s^i_{uw}$.

$\underline{j}A_i^T = r\underline{j}$, and so $(\underline{j}A_i^T)(\underline{j}A_i^T)^T = r^2\ell_i$ and (iii)

follows.

(iv) $\sum_{u=1}^{b} \sum_{w=1}^{b} s^i_{uu}\, s^j_{ww} = \sum_{u=1}^{b} s^i_{uu} \sum_{w=1}^{b} s^j_{ww} = r\ell_j \sum_{u=1}^{b} s^i_{uu} = r^2\ell_i\ell_j$

(using part (i)). ¤

<u>Lemma 2.1.5</u> Suppose $\underline{S}$ admits a P Division $\underline{P}_1,\ldots,\underline{P}_d$ with connection numbers λ_{ij} $(1\le i,j\le d)$, and $\underline{S}^*$ admits a P Division $\underline{B}_1,\ldots,\underline{B}_c$, $|\underline{B}_j| = m_j$ with connection numbers ρ_{ij} $(1\le i,j\le c)$. (We will use this notation throughout for P Divisions of $\underline{S}^*$.) Then, if $x_{u1},\ldots,x_{um_u}$ is a labelling of the blocks of $\underline{B}_u$, and $s^{i}_{(ue)(wf)} = |x_{ue} \cap x_{wf} \cap \underline{P}_i|$,

$$\sum_{i=1}^{d}\sum_{j=1}^{d}\lambda_{ij}\, s^{i}_{(ue)(ue)} s^{j}_{(wf)(wf)} + \sum_{i=1}^{d}(r-\lambda_{ii})\, s^{i}_{(ue)(wf)}$$

$$= \sum_{\substack{j=1\\ j\ne u\\ j\ne w}}^{c} m_j\rho_{ju}\rho_{jw} + \begin{cases} k^2 + (m_u-1)\rho_{uu}^{\,2} & \text{if } u=w, e=f \\ 2k\rho_{uu}+(m_u-2)\rho_{uu}^{\,2} & \text{if } u=w, e\ne f \\ 2k\rho_{uw}+(m_u-1)\rho_{uu}\rho_{uw}+(m_w-1)\rho_{ww}\rho_{uw} & \text{if } u\ne w \end{cases}.$$

<u>Proof</u> Suppose A is an incidence matrix for $\underline{S}$ associated with both $\underline{P}_1,\ldots,\underline{P}_d$ and $\underline{B}_1,\ldots,\underline{B}_c$. Then consider the matrix identity : $(A^TA)^2 = A^T(AA^T)A$.

$$A^TA = \begin{array}{c} \\ \updownarrow m_1 \\ \\ \updownarrow m_2 \\ \\ \vdots \end{array} \begin{array}{c} \leftarrow m_1 \rightarrow \quad \leftarrow m_2 \rightarrow \quad \cdots \\ \left(\begin{array}{c|c|c} \begin{matrix} k & & \rho_{11} \\ & k \ddots & \\ \rho_{11} & & k \end{matrix} & \rho_{12} & \cdots \\ \hline \rho_{12} & \begin{matrix} k & & \rho_{22} \\ & k \ddots & \\ \rho_{22} & & k \end{matrix} & \cdots \\ \hline \vdots & \vdots & \ddots \end{array}\right) \end{array}$$

and so the $\sum_{t=1}^{u-1} m_t + e,\ \sum_{t=1}^{w-1} m_t + f$ entry of $(A^TA)^2$

(i.e. the entry in row $\sum_{t=1}^{u-1} m_t + e$ and column $\sum_{t=1}^{w-1} m_t + f$)

$$= \left\{ \begin{array}{ll} \sum\limits_{\substack{j=1 \\ j\neq u}}^{c} m_j \rho_{ju}^2 + k^2 + (m_u - 1)\rho_{uu}^2 & \text{if } u=w,\ e=f \\[2ex] \sum\limits_{\substack{j=1 \\ j\neq u}}^{c} m_j \rho_{ju}^2 + 2k\rho_{uu} + (m_u - 2)\rho_{uu}^2 & \text{if } u=w,\ e\neq f \\[2ex] \sum\limits_{\substack{j=1 \\ j\neq u \text{ or } w}}^{c} m_j \rho_{ju}\rho_{jw} + 2k\rho_{uw} + (m_u - 1)\rho_{uu}\rho_{uw} + (m_w - 1)\rho_{ww}\rho_{uw} & \text{if } u\neq w \end{array} \right\}.$$

Now $AA^T = X + Y$, where

$$X = \begin{array}{c} \\ \updownarrow \ell_1 \\ \updownarrow \ell_2 \\ \vdots \end{array} \begin{array}{c} \leftarrow \ell_1 \rightarrow \quad \leftarrow \ell_2 \rightarrow \quad \cdots \\ \left(\begin{array}{c|c|c} \lambda_{11} & \lambda_{21} & \cdots \\ \hline \lambda_{12} & \lambda_{22} & \cdots \\ \hline \vdots & \vdots & \ddots \end{array} \right) \end{array}$$

$$Y = \begin{array}{c} \\ \updownarrow \ell_1 \\ \updownarrow \ell_2 \\ \vdots \end{array} \begin{array}{c} \leftarrow \ell_1 \rightarrow \quad \leftarrow \ell_2 \rightarrow \quad \cdots \\ \left(\begin{array}{c|c|c} \begin{matrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_1 \end{matrix} & 0 & \cdots \\ \hline 0 & \begin{matrix} \alpha_2 & & 0 \\ & \ddots & \\ 0 & & \alpha_2 \end{matrix} & \cdots \\ \hline \vdots & \vdots & \ddots \end{array} \right) \end{array}$$

where $\alpha_i = r - \lambda_{ii}$.

So $A^T(AA^T)A = A^TXA + A^TYA$.

Let $A = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_d \end{pmatrix}$ as in the proof of Lemma 2.1.4.

Then

$$YA = \begin{pmatrix} (r-\lambda_{11})A_1 \\ (r-\lambda_{22})A_2 \\ \vdots \\ (r-\lambda_{dd})A_d \end{pmatrix}, \text{ and hence}$$

$$A^T YA = \sum_{i=1}^{d} (r-\lambda_{ii}) A_i^T A_i$$

and so the $\sum_{t=1}^{u-1} m_t+e, \sum_{t=1}^{w-1} m_t+f$ entry of $A^T YA$

$$= \sum_{i=1}^{d} (r-\lambda_{ii})\, s^{i}_{(ue)(wf)}.$$

$$XA = \begin{pmatrix} \sum_{j=1}^{d} \lambda_{1j} J_{\ell_1,\ell_j} A_j \\ \sum_{j=1}^{d} \lambda_{2j} J_{\ell_2,\ell_j} A_j \\ \vdots \\ \sum_{j=1}^{d} \lambda_{dj} J_{\ell_d,\ell_j} A_j \end{pmatrix}$$

where $J_{a,b}$ is the $a \times b$ matrix, every entry of which is +1.

So $A^T XA = \sum_{i=1}^{d} \sum_{j=1}^{d} \lambda_{ij} A_i^T J_{\ell_i,\ell_j} A_j$,

and the $\sum_{t=1}^{u=1} m_t+e, \sum_{t=1}^{w-1} m_t+f$ entry of $A^T XA$

$= \sum_{i=1}^{d} \sum_{j=1}^{d} \lambda_{ij}\, s^{i}_{(ue)(ue)},\, s^{j}_{(wf)(wf)}$. The result now follows. ¤

<u>Lemma 2.1.6</u> If $\underline{S}$ admits a P Division $\underline{P}_1,\ldots,\underline{P}_d$, then :-

(i) $\sum_{\substack{j=1\\ j\neq i}}^{d} \ell_j\lambda_{ij} + (\ell_i-1)\lambda_{ii} = r(k-1)$ for every i $(1\leq i\leq d)$.

If $\underline{S}^*$ also admits a P Division, $\underline{B}_1,\ldots,\underline{B}_c$, then :-

(ii) $\sum_{\substack{j=1\\ j\neq i}}^{c} m_j\rho_{ij} + (m_i-1)\rho_{ii} = k(r-1)$ for every i $(1\leq i\leq c)$;

and (iii) $\sum_{i=1}^{c}\sum_{\substack{j=1\\ j\neq i}}^{c} m_i m_j\rho_{ij}^{\ 2} + \sum_{i=1}^{c} m_i(m_i-1)\rho_{ii}^{\ 2}$

$$= \sum_{i=1}^{d}\sum_{\substack{j=1\\ j\neq i}}^{d} \ell_i\ell_j\lambda_{ij}^{\ 2} + \sum_{i=1}^{d} \ell_i(\ell_i-1)\lambda_{ii}^{\ 2} + vr(r-k).$$

<u>Proof</u> (i) Let A be an incidence matrix for $\underline{S}$, associated with the P Division $\underline{P}_1,\ldots,\underline{P}_d$, and let $A = \begin{pmatrix} A_1\\ A_2\\ \cdot\\ \cdot\\ A_d\end{pmatrix}$ as above.

Then consider the matrix identity :

$$\underline{j}(AA_i^{\ T}) = (\underline{j}A)A_i^{\ T}$$

$$AA_i^T = \begin{pmatrix} A_1A_i^{\ T}\\ A_2A_i^{\ T}\\ \cdot\\ \cdot\\ A_dA_i^{\ T}\end{pmatrix}, \text{ and so } \underline{j}(AA_i^{\ T}) = \sum_{j=1}^{d}\underline{j}A_jA_i^{\ T}$$

$$= \Big(\sum_{j=1}^{d}\ell_j\lambda_{ij} + (r-\lambda_{ii})\Big)\underline{j}\ .$$

$\underline{j}A = k\underline{j}$, and so $(\underline{j}A)A_i^{\ T} = k\underline{j}A_i^{\ T} = kr\underline{j}$, and (i) follows.

(ii) Is just the dual result to (i).

(iii) By Lemma 2.1.5 :-

$$\sum_{i=1}^{d} \sum_{j=1}^{d} \lambda_{ij} s_{\binom{i}{ue}(ue)} s_{\binom{j}{ue}(ue)} + \sum_{i=1}^{d} (r-\lambda_{ii}) s_{\binom{i}{ue}(ue)}$$

$$= \sum_{\substack{j=1 \\ j \neq u}}^{c} m_j \rho_{ju}^{\;2} + k^2 + (m_u - 1)\rho_{uu}^{\;2} .$$

Summing both sides over all blocks x_{ue} of $\underline{S}$, we obtain :-

$$\sum_{i=1}^{d} \sum_{j=1}^{d} \lambda_{ij} \left(\sum_{u=1}^{c} \sum_{e=1}^{m_u} s_{\binom{i}{ue}(ue)} s_{\binom{j}{ue}(ue)} \right)$$

$$+ \sum_{i=1}^{d} (r-\lambda_{ii}) \left(\sum_{u=1}^{c} \sum_{e=1}^{m_u} s_{\binom{i}{ue}(ue)} \right)$$

$$= \sum_{i=1}^{c} m_i \sum_{\substack{j=1 \\ j \neq i}}^{c} m_j \rho_{ij}^{\;2} + bk^2 + \sum_{i=1}^{c} m_i (m_i - 1) \rho_{ii}^{\;2} .$$

Using Lemma 2.1.4 (i) and (ii) we obtain :-

$$\sum_{i=1}^{d} \sum_{\substack{j=1 \\ j \neq i}}^{d} \ell_i \ell_j \lambda_{ij}^{\;2} + \sum_{i=1}^{d} \ell_i (\ell_i - 1) \lambda_{ii}^{\;2} + r \sum_{i=1}^{d} \ell_i \lambda_{ii} + r^2 \sum_{i=1}^{d} \ell_i - r \sum_{i=1}^{d} \ell_i \lambda_{ii}$$

$$= \sum_{i=1}^{c} \sum_{\substack{j=1 \\ j \neq i}}^{c} m_i m_j \rho_{ij}^{\;2} + \sum_{i=1}^{c} m_i (m_i - 1) \rho_{ii}^{\;2} + bk^2$$

and using Corollary 1.1.2(i) and Lemma 2.1.1 , (iii) follows. ¤

<u>Remark</u> Result 1.4.1 (ii) is an immediate corollary of (i).

Lemma 2.1.7 Suppose $\underline{P}_1,\ldots,\underline{P}_d$ is a P Division of $\underline{S}$, and $\underline{B}_1,\ldots,\underline{B}_c$ is a P Division of $\underline{S}^*$. Label the points of $\underline{P}_i$: $P_{i1},\ldots,P_{i\ell_i}$, and label the blocks of $\underline{B}_j$: $x_{j1},\ldots,x_{jm_j}$. Suppose also that there are $\beta_{i(jt)}$ points of $\underline{P}_i$ incident with block x_{jt} of $\underline{B}_j$ and $\gamma_{(is)j}$ blocks of $\underline{B}_j$ incident with point P_{is} of $\underline{P}_i$. (Note that if $\underline{P}_1,\ldots,\underline{P}_d$ and $\underline{B}_1,\ldots,\underline{B}_c$ form a tactical decomposition of $\underline{S}$, $\beta_{i(jt)} = \beta_{ij}$ and $\gamma_{(is)_j} = \gamma_{ij}$ in the notation of Section 1.5, for every t and s.)

Then, for all choices of P_{is} and x_{jt} :-

$$\sum_{u=1}^{d} \lambda_{ui}\beta_{u(jt)} = \sum_{w=1}^{c} \rho_{wj}\ \gamma_{(is)w} + (k-r+\lambda_{ii}-\rho_{jj})\delta$$

$$\text{where } \delta = \left\{ \begin{array}{l} 1 \text{ if } P_{is} \text{ is incident with } x_{jt} \\ \\ 0 \text{ otherwise} \end{array} \right\}.$$

Proof Let A be an incidence matrix for $\underline{S}$ associated with the P Divisions of $\underline{S}$ and $\underline{S}^*$. Consider the matrix identity :

$$A(A^TA) = (AA^T)A.$$

The $\sum_{w=1}^{i-1} \ell_w+s,\ \sum_{w=1}^{j-1} m_w+t$ entry of $A(A^TA)$

$$= \sum_{w=1}^{c} \rho_{wj}\ \gamma_{(is)w} + (k-\rho_{jj})\delta$$

and the $\sum_{w=1}^{i-1} \ell_w+s,\ \sum_{w=1}^{j-1} m_w+t$ entry of $(AA^T)A$

$$= \sum_{u=1}^{d} \lambda_{ui}\beta_{u(jt)} + (r-\lambda_{ii})\delta;$$ and the Lemma follows. ⌑

We now generalize Result 1.4.1 (iii),(iv) to obtain results on the incidence matrices of 1-structures admitting P Divisions.

<u>Lemma 2.1.8</u> Suppose $\underline{S}$ admits a P Division $\underline{P}_1,\ldots,\underline{P}_d$. Let A be an incidence matrix for $\underline{S}$ associated with $\underline{P}_1,\ldots,\underline{P}_d$. Then AA^T has among its eigenvalues rk and $r-\lambda_{ii}$ $(1\le i\le d)$, with multiplicities at least 1 and ℓ_i-1 $(1\le i\le d)$ respectively.

<u>Proof</u> By inspection, $\underline{j}$ and $\underline{e}_u$ $(1\le u\le v-1;\ u\ne \sum_{j=1}^{t}\ell_j$ for any $t)$ are eigenvectors of AA^T with the appropriate eigenvalues. ¤

<u>Corollary 2.1.9</u> Suppose $\underline{D}$ admits a P Division $\underline{P}_1,\ldots,\underline{P}_d$.

Then A^TA (where A is an incidence matrix for $\underline{D}$) has among its eigenvalues :- rk, $r-\lambda_{11}$, $r-\lambda_{22}$, ..., $r-\lambda_{dd}$ with multiplicities at least $1, \ell_1-1, \ell_2-1, \ldots, \ell_d-1$.

<u>Proof</u> Since $\underline{D}$ is a 1-design $r > \lambda_{ii}$ for every i. Hence, using the fact that AA^T and A^TA have the same non-zero eigenvalues, Lemma 2.1.8 gives the Corollary. ¤

<u>Result 2.1.10 (Adhikary, [1])</u> Suppose $\underline{S}$ admits a P Division $\underline{P}_1,\ldots,\underline{P}_d$. Let A be an incidence matrix for $\underline{S}$. Then :-

$$|AA^T| = |B| . \prod_{i=1}^{d} (r-\lambda_{ii})^{\ell_i - 1}, \quad \text{where } B = (b_{ij}) \text{ is a}$$

$d\times d$ matrix with $b_{ij} = \ell_j\lambda_{ij} + \delta_{ij}\,(r-\lambda_{ii})$.

2.2 Generalisations of some Results of Bose and Connor

We now generalise several well-known results on GD designs to structures admitting a P Division.

<u>Theorem 2.2.1</u> If $\underline{S}$ admits a point division $\underline{P}_1,\ldots,\underline{P}_d$ then:-

(i) $(v-\ell_i)rk \geq v\sum_{\substack{j=1\\j\neq i}}^{d} \ell_j\lambda_{ij}$ for every i $(1\leq i\leq d)$; with equality for a given i, if and only if every block of $\underline{S}$ contains equally many points of $\underline{P}_i$; and

(ii) If $d>1$ and $\underline{P}_i$ is such that $(v-\ell_i)rk = v\sum_{\substack{j=1\\j\neq i}}^{d} \ell_j\lambda_{ij}$, then $1<\ell_i<v$, $r>\lambda_{ii}$, and $\left(\sum_{\substack{j=1\\j\neq i}}^{d} \ell_j\lambda_{ij}\right)/(v-\ell_i) > \lambda_{ii}$.

<u>Proof</u>

(i) Using the notation of Lemma 2.1.4 :

$$\sum_{u=1}^{b}\left(s^i_{uu}-r\ell_i/b\right)^2 = \sum_{u=1}^{b}\left(s^i_{uu}\right)^2 - 2r\ell_i\left(\sum_{u=1}^{b} s^i_{uu}\right)/b + r^2\ell_i{}^2b/b^2$$

$$= \ell_i{}^2\lambda_{ii}+\ell_i(r-\lambda_{ii})-2r^2\ell_i{}^2/b+r^2\ell_i{}^2/b \quad \text{(by Lemma 2.1.4(i)(ii))}$$

$$= \ell_i\left(v((\ell_i-1)\lambda_{ii}+r)-rk\ell_i\right)/v \quad \text{(by Corollary 1.1.2(i))}$$

$$= \ell_i\left((v-\ell_i)rk - v\sum_{\substack{j=1\\j\neq i}}^{d}\ell_j\lambda_{ij}\right) \quad \text{(by Lemma 2.1.6(i))}$$

L.H.S. ≥ 0 and so $(v-\ell_i)rk \geq v\sum_{\substack{j=1\\j\neq i}}^{d}\ell_j\lambda_{ij}$.

$(v-\ell_i)rk = v\sum_{\substack{j=1\\j\neq i}}^{d}\ell_j\lambda_{ij}$ if and only if $s^i_{uu} = r\ell_i/b$ for every u, and (i) follows.

(ii) Firstly, $v>\ell_i$ since $d>1$. By assumption we have :

$$(v-\ell_i)rk = v\sum_{\substack{j=1\\j\neq i}}^{d}\ell_j\lambda_{ij}, \text{ and by Lemma 2.1.6(i):}$$

$$\sum_{\substack{j=1\\j\neq i}}^{d}\ell_j\lambda_{ij} = r(k-1) - (\ell_i-1)\lambda_{ii}. \quad \text{Hence } \ell_i(rk-v\lambda_{ii})$$

$= v(r-\lambda_{ii})$. So, since $\ell_i \geq 1$, $v(r-\lambda_{ii}) = 0$ if and only if $rk-v\lambda_{ii} = 0$. But $vr \neq kr$ and hence we always have $v(r-\lambda_{ii}) \neq 0 \neq rk-v\lambda_{ii}$. Using this we have $\ell_i = (vr-v\lambda_{ii})/(kr-v\lambda_{ii}) > 1$, and since $v(r-\lambda_{ii}) \neq 0$, $r>\lambda_{ii}$.

Also $v\sum_{\substack{j=1\\j\neq i}}^{d}\ell_j\lambda_{ij} = (v-\ell_i)rk$

$= (v-\ell_i)\left(r+(\ell_i-1)\lambda_{ii} + \sum_{\substack{j=1\\j\neq i}}^{d}\ell_j\lambda_{ij}\right)$, and so

$$(r-\lambda_{ii})(v-\ell_i) = \ell_i\left(\sum_{\substack{j=1\\j\neq i}}^{d}\ell_j\lambda_{ij}-(v-\ell_i)\lambda_{ii}\right).$$

$v-\ell_i>0$, and so the L.H.S. > 0 and (ii) follows. ¤

As Corollaries we have the following results of Bose and Connor, originally stated in Chapter 1.

Result 1.4.3 If $\underline{D}$ is GD then $rk \geq v\lambda'$

Proof From (i) above : $(v-\ell_i)rk \geq v\sum_{\substack{j=1\\j\neq i}}^{d}\ell_j\lambda_{ij}$

But for a GD design, $\ell_i=\ell$, $\lambda_{ij}=\lambda'$ $(i\neq j)$. ¤

Result 1.4.5 If $\underline{D}$ is GD then $\underline{D}$ is SRGD if and only if every block is incident with precisely k/d points of each point class.

Proof $\underline{D}$ is SRGD if and only if $rk=v\lambda'$ (by definition), and applying (i), the result follows. ¤

Result 1.4.6 If $\underline{D}$ is SRGD then $\lambda<\lambda'$

Proof Immediate from (ii) above. ¤

Because of the analagous natures of Theorem 2.2.1 and Results 1.4.5 and 1.4.6 we make the following definition.

If $\underline{P}_1,\ldots,\underline{P}_d$ is a P Division of $\underline{S}$, then $\underline{P}_1,\ldots,\underline{P}_d$ is said to be a Semi-regular P Division (SRP Division) if

$$(v-\ell_i)rk = v\sum_{\substack{j=1\\ j\neq i}}^{d} \ell_j\lambda_{ij} \text{ for every } i,\ (1\leq i\leq d).$$

As we shall see in section 2.3, this generalizes the notion of semi-regular group divisibility. We now have :-

Lemma 2.2.2 If $\underline{P}_1,\ldots,\underline{P}_d$ is a SRP Division of $\underline{S}$ it is maximal and hence unique.

Proof By Theorem 2.2.1 (i), if $\underline{Q}_1,\ldots,\underline{Q}_e$ is a P Division having $\underline{P}_1,\ldots,\underline{P}_d$ as a refinement, then $\underline{Q}_1,\ldots,\underline{Q}_e$ is a SRP Division. By Lemma 2.1.6(i) and the definitions of SRP Division we see that $\ell_i = v(r-\lambda_{ii})/(rk-v\lambda_{ii})$ for every $i(1\leq i\leq d)$, i.e. the size of a point class in a SRP Division of $\underline{S}$ depends on the connection number of two points within it. Hence $\{\underline{Q}_1,\ldots,\underline{Q}_e\} = \{\underline{P}_1,\ldots,\underline{P}_d\}$. So $\underline{P}_1,\ldots,\underline{P}_d$ is maximal, and hence unique by Lemma 2.1.3. ¤

Because of Lemma 2.2.2 we call a structure admitting a SRP Division, Semi-regular point divisible (SRPD), and if $\underline{S}$ is SRPD, we will always assume (unless otherwise stated) that

$\underline{P}_1,\ldots,\underline{P}_d$ with connection numbers λ_{ij} is the SRP Division associated with $\underline{S}$.

Remark Any 2-design admits a P Division with $d = 1$, and clearly this is the unique SRP Division of a 2-design;

$(v-\ell_i) = 0 = \sum_{\substack{j=1\\j\neq i}}^{d} \ell_j\lambda_{ij}$, and so equality holds trivially in

Theorem 2.2.1(i).

Theorem 2.2.3 Suppose $\underline{D}$ admits a P Division $\underline{P}_1,\ldots,\underline{P}_d$. Then :-

(i) $b + d \geq v + 1$;

(ii) If $\underline{P}_1,\ldots,\underline{P}_d$ is a CLP Division and $b + d = v + 1$, then $\underline{D}^*$ is a 2-(b,r,ρ) design, where $\rho = k-r+\lambda$;

and (iii) Suppose there exist constants λ_1,λ_2 with $\lambda_{ii} = \lambda_1$ or λ_2 for every i with $\ell_i>1$ and also suppose $\underline{D}$ has two intersection numbers. Then if $b + d = v + 1$, $\underline{D}^*$ is a PBD(2).

Proof (i) Since $\underline{D}$ is a 1-design, $r>\lambda_{ii}$ for every i, and so, by Lemma 2.1.8, AA^T has $v-d+1$ distinct eigenvectors with non-zero eigenvalues. So $v-d+1 \leq \text{rank } AA^T = \text{rank } A \leq b$; hence (i).

(ii) If $b = v - d + 1$, then, by Corollary 2.1.9, A^TA has eigenvalues rk, $r-\lambda_{11}$, $r-\lambda_{22},\ldots,r-\lambda_{dd}$ with multiplicities : $1,\ell_1-1,\ell_2-1,\ldots,\ell_d-1$. Hence if $\underline{P}_1,\ldots,\underline{P}_d$ is a CLP Division, then $A^TA|\underline{j}^{\perp}$ has just one eigenvalue : $r-\lambda$, and so, by Lemma 1.2.4, $\underline{D}^*$ is a 2-(b,r,ρ) design and $k-\rho = r-\lambda$.

(iii) As for (ii), if $b = v-d+1$, then $\Lambda^T\Lambda|\underline{j}^{\perp}$ has just two eigenvalues: $r-\lambda_1$ and $r-\lambda_2$. Hence, since $\underline{D}^*$ has two connection numbers, (iii) follows from Lemma 1.4.11. ¤

Remark A stronger result than (ii) exists, see Theorem 2.5.7 below.

Finally, we obtain a bound on the intersection numbers of certain designs admitting a P Division, generalizing a result on SRGD designs due to Saraf, [37].

Lemma 2.2.4 If $\underline{S}$ admits a P Division $\underline{P}_1,\ldots,\underline{P}_d$; then, with the notation of Lemma 2.1.4 :-

$$\sum_{t=1}^{b} s_{ut}\, s_{wt} = \sum_{i=1}^{d}\sum_{j=1}^{d}\lambda_{ij}\, s^i_{uu}s^j_{ww} + \sum_{i=1}^{d}(r-\lambda_{ii})\, s^i_{uw}; \quad \text{for every}$$

u,w $(1\le u,w\le b)$; where $s_{uw} = |x_u \cap x_w| = \sum_{i=1}^{d} s^i_{uw}$.

Proof Consider the trivial P Division of $\underline{S}^*$, and then, by Lemma 2.1.5, we have :-

$$\sum_{i=1}^{d}\sum_{j=1}^{d}\lambda_{ij}\, s^i_{uu}s^j_{ww} + \sum_{i=1}^{d}(r-\lambda_{ii})s^i_{uw} = \sum_{\substack{t=1\\ t\ne u\\ t\ne w}}^{b} s_{ut}s_{wt} + \begin{cases} k^2 & \text{if } u = w \\ 2ks_{uw} & \text{if } u \ne w \end{cases}.$$

The Lemma now follows. ¤

Lemma 2.2.5 If $\underline{S}$ admits a P Division $\underline{P}_1,\ldots,\underline{P}_d$, then, with the above notation :

$$\sum_{t=1}^{b}(s_{ut}-s_{wt})^2 = \sum_{i=1}^{d}\sum_{j=1}^{d}\lambda_{ij}(s^i_{uu}s^j_{uu}-2s^i_{uu}s^j_{ww}+ s^i_{ww}s^j_{ww})$$

$$+ \sum_{i=1}^{d}(r-\lambda_{ii})(s^i_{uu}-2s^i_{uw}+s^i_{ww})$$

for every u,w $(1\le u,w\le b)$.

Proof $\sum_{t=1}^{b}(s_{ut}-s_{wt})^2 = \sum_{t=1}^{b} s_{ut}^2 - 2\sum_{t=1}^{b} s_{ut}s_{wt} + \sum_{t=1}^{b} s_{wt}^2$, and,

applying Lemma 2.2.4, the result follows.¤

Theorem 2.2.6 Suppose $\underline{D}$ admits a CLP Division $\underline{P}_1,\ldots,\underline{P}_d$, and, (using the above notation), x_u, x_w are two distinct blocks satisfying $s^i_{uu} = s^i_{ww}$ for every i ($1 \le i \le d$).

Then $s_{uw} \ge k-r+\lambda$ with equality if and only if $s_{tu} = s_{tw}$ for every t ($1 \le t \le w, t \ne u, t \ne w$).

Proof Lemma 2.2.5 gives :-

$$\sum_{t=1}^{b}(s_{ut}-s_{wt})^2 = \sum_{i=1}^{d}\sum_{j=1}^{d}\lambda_{ij}(s^i_{uu}s^j_{uu}-2s^i_{uu}s^j_{ww}+s^i_{ww}s^j_{ww})$$

$$+ (r-\lambda)\sum_{i=1}^{d}(s^i_{uu}-2s^i_{uw}+s^i_{ww})$$

$$= 2(r-\lambda)(k-s_{uw}), \text{ since } s^i_{uu} = s^i_{ww} \text{ for every } i.$$

Hence $$\sum_{\substack{t=1\\ t\ne u\\ t\ne w}}^{b}(s_{ut}-s_{wt})^2 = 2(r-\lambda)(k-s_{uw})-2(k-s_{uw})^2$$

$$= 2\left(s_{uw}-(k-r+\lambda)\right)(k-s_{uw}).$$

L.H.S. ≥ 0 and $k > s_{uw}$, by definition of design.

Hence $s_{uw} \ge (k-r+\lambda)$.

$s_{uw} = k-r+\lambda$ if and only if R.H.S. = 0, i.e. if and only if

$s_{ut} = s_{wt}$ for every t. ¤

As a Corollary we have :-

Result 2.2.7 (Saraf, [37]) If $\underline{D}$ is SRGD, and x_u, x_w are two distinct blocks of $\underline{D}$, then :- $s_{uw} \geq k - r + \lambda$ with equality if and only if $s_{ut} = s_{wt}$ for every t ($t \neq u$ or w).

2.3 Semi-regular Point Divisible Structures

We now consider SRPD structures and generalize some results for SRGD designs.

Lemma 2.3.1 If $\underline{S}$ is SRPD then $b + d \geq v + 1$.

Proof By Theorem 2.2.1 $r > \lambda_{ii}$ for every i ($1 \leq i \leq d$); and so, by Lemma 2.1.8, AA^T has $v-d+1$ distinct eigenvectors with non-zero eigenvalues, where A is an incidence matrix for $\underline{S}$. Hence $v-d+1 \leq \text{rank } AA^T = \text{rank } A \leq b$. ¤

Remark Note that this is a stronger result than Theorem 2.2.3(i), since there we needed to assume $r-\lambda_{ii} > 0$ for every i.

Lemma 2.3.2 If $\underline{S}$ admits a P Division $\underline{P}_1, \ldots, \underline{P}_d$ and

$$(v-\ell_i)rk = v\sum_{\substack{j=1 \\ j \neq i}}^{d} \ell_j \lambda_{ij} \quad \text{for every } i \neq u \text{ for some } u, \text{ then } \underline{P}_1, \ldots, \underline{P}_d$$

is a SRP Division.

Proof By Theorem 2.2.1(i), every block is incident with $r\ell_i/b$ blocks of $\underline{P}_i$ (for every $i \neq u$). Hence every block is incident with $k - \sum_{\substack{i=1 \\ i \neq u}}^{d} r\ell_i/b = r\ell_u/b$ (by Corollary 1.1.2(i) and Lemma 2.1.1) blocks of $\underline{P}_u$. So using Theorem 2.2.1(i) the Lemma follows. ¤

<u>Lemma 2.3.3</u> If $\underline{S}$ is SRPD then :-

$$\text{(i)} \quad \lambda_{ii} = \frac{(\ell_i-\ell_j)r + \ell_i(\ell_j-1)\lambda_{ij}}{\ell_j(\ell_i-1)} \quad \text{for every } i,j;$$

and (ii) there exists a λ' such that $\lambda_{ij} = \lambda'$ for every i,j $(i \neq j)$.

<u>Proof</u> Let A be an incidence matrix for $\underline{S}$ associated with $\underline{P}_1,\ldots,\underline{P}_d$. Put

$$\underline{f}_{ij} = \Big(\overbrace{0\,0\,\ldots\,0}^{\sum_{u=1}^{i-1}\ell_u}\ \overbrace{\ell_i^{-1}\,\ell_i^{-1}\,\ldots\,\ell_i^{-1}}^{\ell_i}\ \overbrace{0\,0\,\ldots\,0}^{\sum_{u=i+1}^{j-1}\ell_u}\ \overbrace{-\ell_j^{-1}\,-\ell_j^{-1}\ldots\,-\ell_j^{-1}}^{\ell_j}\ \overbrace{0\,0\,\ldots\,0}^{\sum_{u=j+1}^{d}\ell_u}\Big).$$

Then $\underline{f}_{ij}\,A = \underline{0}$ for every i,j, since every block is incident with precisely $r\ell_i/b$ points of $\underline{P}_i$, by Theorem 2.2.1(i).

Hence $\underline{f}_{ij}\,AA^T = \underline{0}$ for every i,j. From this we may derive the following identities :-

$\big(r+(\ell_i-1)\lambda_{ii}\big)/\ell_i = \lambda_{ij}$; $\big(r+(\ell_j-1)\lambda_{jj}\big)/\ell_j = \lambda_{ij}$; and $\lambda_{iw} = \lambda_{jw}$ for every w $(w \neq i, w \neq j)$, and the Lemma follows. ¤

<u>Remark</u> Because of (ii) above we will denote the connection numbers of a SRPD 1-structure by λ_{ii} $(1 \le i \le d)$ and λ'.

<u>Corollary 2.3.4</u> If $\underline{S}$ is SRPD then $\lambda_{ii} < \lambda'$ for every i $(1 \le i \le d)$.

<u>Proof</u> By Theorem 2.2.1(ii) :

$$\Big(\sum_{\substack{j=1\\ j\neq i}}^{d} \ell_j\lambda_{ij}\Big)/(v-\ell_i) > \lambda_{ii}. \qquad \lambda_{ij} = \lambda' \text{ for every } i,j\ (i\neq j)$$

by Lemma 2.3.3, and so $\Big(\lambda'\sum_{\substack{j=1\\ j\neq i}}^{d} \ell_j\Big)/(v-\ell_i) > \lambda_{ii}$. Applying Lemma 2.1.1. the Corollary follows. ¤

Theorem 2.3.5 If $\underline{P}_1,\ldots,\underline{P}_d$ is a SRP Division of $\underline{S}$, then the following are equivalent :-

(i) $\underline{P}_1,\ldots,\underline{P}_d$ is a semi-regular group division.

(ii) There exists a constant λ such that $\lambda_{ii} = \lambda$ for every i;

(iii) $|\underline{P}_i| = \ell = v/d$ for every i $(1 \leq i \leq d)$.

Proof (i) $\Rightarrow$ (ii) : By definition.

(ii) $\Rightarrow$ (iii) : From Theorem 2.2.1(ii) :-

$\ell_i = v(r-\lambda_{ii})/(rk-v\lambda_{ii}) = v(r-\lambda)/(rk-v\lambda)$; i.e. ℓ_i is a constant for every i. Hence $\ell_i = v/d = \ell$.

(iii) $\Rightarrow$ (i) : By Lemma 2.3.3(i) $\underline{P}_1,\ldots,\underline{P}_d$ is a CLP Division, and hence a group division; (by Lemma 2.3.3 (ii)). Finally $\underline{P}_1,\ldots,\underline{P}_d$ is semi-regular by Result 1.4.5. ¤

Remark There do exist SRP Divisions $\underline{P}_1,\ldots,\underline{P}_d$ of $\underline{S}$ which are not group divisions. See, for example, Adhikary, [1], p.81, example 1.

2.4 Point Divisions and Tactical Decompositions

Recall from section 2.1, that a Tactical Division is simply a Tactical Decomposition whose point classes form a CLP Division. To study Tactical Divisions we first require the following combinatorial identities.

Result 2.4.1 If $T(\underline{S})$ is a tactical decomposition of $\underline{S}$, then :-

(i) $\sum_{j=1}^{c} \gamma_{ij} = r$, for every i $(1 \leq i \leq d)$;

(ii) $\sum_{i=1}^{d} \beta_{ij} = k$, for every j $(1 \leq j \leq c)$;

(iii) $\ell_i\gamma_{ij} = m_j\beta_{ij}$, for every i,j $(1 \leq i \leq d, 1 \leq j \leq c)$.

Proof Trivial. ¤

Lemmas 2.4.2 and 2.4.3 and Theorem 2.4.4 below are due to Beker, [6]; we present them here using new proofs based on results obtained above. The proof of Theorem 2.4.4 is essentially the same as that used in proving Theorem 1 of Beker, [5].

Lemma 2.4.2 Suppose $T(\underline{S})$ is a tactical decomposition of $\underline{S}$. If the point classes of $T(\underline{S})$ form a P Division of $\underline{S}$, then :

(i) $\sum_{u=1}^{c} m_u \beta_{iu} \beta_{ju} = \ell_i \ell_j \lambda_{ij} + \ell_i (r-\lambda_{ii}) \delta_{ij}$ for every i,j $(1 \leq i,j \leq d)$;

(ii) $\sum_{u=1}^{c} \gamma_{iu} \beta_{ju} = \ell_j \lambda_{ij} + (r-\lambda_{ii}) \delta_{ij}$ for every i,j $(1 \leq i,j \leq d)$;

(iii) $\sum_{u=1}^{c} \gamma_{iu} \gamma_{ju} / m_u = \lambda_{ij} + (r-\lambda_{ii}) \delta_{ij} / \ell_i$ for every i,j $(1 \leq i,j \leq d)$.

Dually, if the block classes of $T(\underline{S})$ form a P Division of $\underline{S}^*$, then:-

(iv) $\sum_{u=1}^{d} \ell_u \gamma_{ui} \gamma_{uj} = m_i m_j \rho_{ij} + m_i (k-\rho_{ii}) \delta_{ij}$ for every i,j $(1 \leq i,j \leq c)$;

(v) $\sum_{u=1}^{d} \beta_{ui} \gamma_{uj} = m_j \rho_{ij} + (k-\rho_{ii}) \delta_{ij}$ for every i,j $(1 \leq i,j \leq c)$;

(vi) $\sum_{u=1}^{d} \beta_{ui} \beta_{uj} / \ell_u = \rho_{ij} + (k-\rho_{ii}) \delta_{ij} / m_i$ for every i,j $(1 \leq i,j \leq c)$.

Where δ_{ij} = the Kronecker Delta.

Proof (i) Immediate from Lemma 2.1.4(ii).

(ii) and (iii) follow from (i), using Result 2.4.1(iii).

(iv) - (vi) are just the dual results of (i) - (iii). ¤

Lemma 2.4.3 If T(D) is a tactical division of D, and x,y are two distinct blocks from the same block class of T(D), then $|x \cap y| \geq k-r+\lambda$, with equality if and only if $|x \cap z| = |y \cap z|$ for every block z ($z \neq x$ or y).

Proof Immediate from Theorem 2.2.6. ¤

Theorem 2.4.4 Suppose T(D) is a tactical division of D. Then :-

(i) $b+d \geq v+c$;

(ii) The following are equivalent :

(a) $b+d = v+c$ (i.e. T(D) is strong);

(b) The block classes of T(D) form a point division of D*;

(c) Every two distinct blocks from the same block class of T(D) intersect in $k-r+\lambda$ points.

Proof (i) Set $S = \sum_{i=1}^{d} \sum_{j=1}^{c} \beta_{ij}\gamma_{ij}$

By Lemma 2.4.2 (ii), $S = \sum_{i=1}^{d} (\ell_i\lambda+(r-\lambda))$

$= \lambda v + (r-\lambda)d$ (by Lemma 2.1.1).

Pick $x_j \in B_j$ for every j ($1 \leq j \leq c$), and count triples (x,x_j,P) (where PIx, PIx_j and $x \in B_j$) to obtain :

$$S = \sum_{j=1}^{c} \sum_{x \in B_j} |x \cap x_j|$$

Set $\bar{\rho} = \left(\sum_{j=1}^{c} \sum_{\substack{x \in \underline{B}_j \\ x \neq x_j}} |x \cap x_j|\right)/(b-c)$.

So $(k-\bar{\rho})c + \bar{\rho}b = S$.

So, using Corollary 1.1.2(i), we have :

$$(k-\bar{\rho})(b-c) = (r-\lambda)(v-d). \qquad (*)$$

All the terms in (*) are positive (since $\underline{D}$ is a design), and, by Lemma 2.4.3, $\bar{\rho} \geq k-r+\lambda$. Hence $b-c \geq v-d$ and (i) follows.

(ii) (b) $\Leftrightarrow$ (c) follow immediately from Lemma 2.4.3.

(a) $\Leftrightarrow$ (c) : By (*), $b+d = v+c$ if and only if $\bar{\rho} = k-r+\lambda$; i.e. if and only if $|x \cap x_j| = k-r+\lambda$ for every $x \in \underline{B}_j - \{x_j\}$, i.e. (by Lemma 2.4.3) if and only if $|x \cap y| = k-r+\lambda$ for every pair of distinct blocks x,y in the same block class of $T(\underline{D})$. ¤

Remark Theorem 2.4.4 is essentially equivalent to Results 1.5.1 and 1.5.2.

Finally we have a generalisation of a result of Shrikhande and Raghavarao, [44], on "σ-resolvable" SRGD designs.

Theorem 2.4.5 If $\underline{D}$ admits a strong tactical division $T(\underline{D})$ with point classes $\underline{P}_1,\ldots,\underline{P}_d$ and block classes $\underline{B}_1,\ldots,\underline{B}_c$; then the following are equivalent :-

(i) $\underline{P}_1,\ldots,\underline{P}_d$ is a semi-regular group division of $\underline{D}$;

(ii) $\underline{P}_1,\ldots,\underline{P}_d$ is a SRP Division of $\underline{D}$;

(iii) $\underline{B}_1,\ldots,\underline{B}_c$ is a semi-regular group division of $\underline{D}^*$;

(iv) $\underline{B}_1,\ldots,\underline{B}_c$ is a SRP Division of $\underline{D}^*$.

Proof (ii) ⇒ (i) Immediate from Theorem 2.3.5.

(i) ⇒ (iv) By Result 1.4.5 every block contains the same number (= k/d) of points from each point class, and clearly every point class contains v/d points.

T(D) is a tactical decomposition and so, by Result 2.4.1(iii), $\ell_i\gamma_{ij} = m_j\beta_{ij}$ for every i,j. Hence $v\gamma_{ij}/d = km_j/d$, i.e. $\gamma_{ij} = km_j/v$ which is independent of the choice of i. (iv) then follows immediately from Theorem 2.2.1(i).

(iv) ⇒ (iii) and (iii) ⇒ (ii) follow by dual arguments. ¤

As a corollary we have :

Result 2.4.6 (Shrikhande and Raghovarao,[44], Corollary 6.1)

An α-resolvable SRGD design D, is affine α-resolvable if and only if b-c = v-d, (where c is the number of classes in the α-resolution ; $c\alpha=r$).

Proof Recall from Section 1.5 that a design D is affine α-resolvable if and only if D* is SRGD, and an α-resolution is a tactical decomposition with one point class such that $\gamma_{1j} = \alpha$ for every j.

If the classes of the group division of D are $P_1,..,P_d$ and the classes of the α-resolution are $B_1,...,B_c$, then $P_1,...,P_d$ and $B_1,...,B_c$ form a tactical decomposition T(D) of D (using Result 1.4.5). So b-c = v-d if and only if T(D) is strong (by definition) if and only if $B_1,...,B_c$ forms a P Division of D* (using Theorem 2.4.4) if and only if $B_1,...,B_c$ is a SRP Division of D* (by Theorem 2.2.1(i)), i.e. if and only if $B_1,...,B_c$ is a semi-regular group division of D (by the above theorem). ¤

Remark So Result 2.4.6 essentially states that (i) and (iii) hold if and only if (i) and (iv) hold in Theorem 2.4.5.

2.5 2-Designs with Divisible Duals

To study 2-designs whose duals admit P Divisions, we first state :

Result 2.5.1 (Majumdar, [32]) If $\underline{D}$ is a 2-design and x,y are two distinct blocks of $\underline{D}$, then $|x \cap y| \geq k-r+\lambda$, with equality if and only if $|x \cap z| = |y \cap z|$ for every block $z (z \neq x \text{ or } y)$.

We now give some definitions from Beker and Haemers, [7]. If $\underline{D}$ is a 2-design and x,y are two blocks of $\underline{D}$, we define $x \sim y$ if and only if $x = y$ or $|x \cap y| = k-r+\lambda$. Then, by Result 2.5.1, $\sim$ is an equivalence relation on the blocks of $\underline{D}$. The partition of the blocks into equivalence classes is called a maximal decomposition of $\underline{D}$, and any refinement of this partition is called a decomposition of $\underline{D}$. Also, if every class of a decomposition contains the same number of blocks, then the decomposition is said to be regular.

It is straightforward to verify that if $\underline{D}$ is a 2-design, then $\underline{B}_1, \ldots, \underline{B}_c$ is a P Division of $\underline{D}^*$ if and only if $\underline{B}_1, \ldots, \underline{B}_c$ is a decomposition of $\underline{D}$, and in this case $\underline{B}_1, \ldots, \underline{B}_c$ is a CLP Division of $\underline{D}^*$. Such designs have been studied extensively in [7].

We call a 2-design Quasi-symmetric if it has precisely two intersection numbers : ρ, ρ', say $(\rho < \rho')$.

Lemma 2.5.2 If $\underline{D}$ is a 2-design and A is an incidence matrix for $\underline{D}$, then $A^T A$ has eigenvalues : rk, $r-\lambda$ and 0 with multiplicities 1, v-1 and b-v.

Proof Immediate from Result 1.2.3(i) and Lemma 1.2.4. ¤

Using Lemmas 1.4.11 and 2.5.2 we immediately have :

Result 2.5.3 (Shrikhande and Bhagwandas, [45]; Goethals and Seidel, [20])

The dual of a quasi-symmetric 2-design is a PBD(2); or, equivalently, the block graphs are strongly regular.

We now reconsider Theorem 2.2.3(ii). As mentioned above, if $\underline{D}$ is a 2-design and $\underline{D}^*$ admits a P Division $\underline{B}_1,\ldots,\underline{B}_c$, then $\underline{B}_1,\ldots,\underline{B}_c$ is a decomposition, and hence a CLP Division with $\rho = k-r+\lambda$.

Result 2.5.4 (Beker and Haemers, [7]) If $\underline{D}$ is a 2-design and $\underline{B}_1,\ldots,\underline{B}_c$ is a decomposition of $\underline{D}$, then:-

$m_j = |\underline{B}_j| \leq b/(b-v+1)$; with equality for every j if and only if $\underline{B}_1,\ldots,\underline{B}_c$ form a resolution of $\underline{D}$.

As a corollary we have :-

Corollary 2.5.5 If $\underline{D}$ is a 2-design and $\underline{B}_1,\ldots,\underline{B}_c$ is a decomposition of $\underline{D}$ with $c = b-v+1$, then $\underline{B}_1,\ldots,\underline{B}_c$ forms a strong resolution of $\underline{D}$.

Proof Consider $\sum_{j=1}^{c}\left(b/(b-v+1)-m_j\right) = b - \sum_{j=1}^{c} m_j = 0$ (since $c = b-v+1$). But every term on the L.H.S. is non-negative by Result 2.5.4, and so $m_j = b/(b-v+1)$ for every j. Hence, by Result 2.5.4 $\underline{B}_1,\ldots,\underline{B}_c$ forms a resolution of $\underline{D}$, which is strong by definition. ¤

Using this corollary in conjunction with an earlier result we now have :

Theorem 2.5.6 If $\underline{B}_1,\ldots,\underline{B}_c$ is a decomposition of a 2-design $\underline{D}$, then $v+c \geq b+1$ with equality if and only if $\underline{B}_1,\ldots,\underline{B}_c$ forms a strong resolution of $\underline{D}$.

Proof Immediate from Theorem 2.2.3(i) and Corollary 2.5.5. ¤

Remark This is a stronger result than a similar Theorem of Beker and Haemers, [7]. They prove the same result with the added assumption that $\underline{B}_1,\ldots,\underline{B}_c$ is regular.

Using this result in conjunction with Theorem 2.2.3(ii), we have the following generalisation of a similar result for SRGD designs.

Theorem 2.5.7 If $\underline{P}_1,\ldots,\underline{P}_d$ is a CLP Division of $\underline{D}$ and $b+d = v+1$, then $\underline{D}^*$ is a $2-(b,r,k-r+\lambda)$ design and $\underline{P}_1,\ldots,\underline{P}_d$ forms a strong resolution of $\underline{D}^*$.

Proof $\underline{D}^*$ is a $2-(b,r,k-r+\lambda)$ design by Theorem 2.2.3(ii), and the Theorem follows by Theorem 2.5.6. ¤

As an immediate corollary we have :-

Result 2.5.8 (Saraf, [37]) If $\underline{D}$ is SRGD and $b = v-d+1$ then $\underline{D}^*$ is a 2-design.

Finally we extend a result due to Beker and Haemers, [7]. In [7], Lemmas 5.2 and 5.3 establish the equivalence of (i),(ii),(iii),(iv) below.

Theorem 2.5.9 If $\underline{D}$ is a Quasi-symmetric 2-design with intersection numbers ρ,ρ' $(\rho<\rho')$, then the following are equivalent :-

(i) $\underline{D}$ is strongly resolvable;

(ii) $\rho = k-r+\lambda$;

(iii) $\rho' = k^2/v$;

(iv) $G(\underline{B},\rho) \cong \Gamma(c,m)$, for some c,m;

(v) $\underline{D}^*$ is GD;

(vi) $\underline{D}^*$ is SRGD;

(vii) $\underline{D}^*$ is SRPD;

(viii) $\underline{D}^*$ admits a P Division with $b+d = v+1$.

Proof (i) $\Rightarrow$ (ii) Immediate from Results 1.5.3 and 2.5.1.

(ii) $\Rightarrow$ (v) Consider the maximal decomposition $\underline{B}_1,\ldots,\underline{B}_c$ of $\underline{D}$. Clearly $\underline{B}_1,\ldots,\underline{B}_c$ is a group division of $\underline{D}^*$, and (v) follows.

(i) $\Rightarrow$(iii) Immediate from Results 1.5.3 and 2.5.1.

(iii) $\Rightarrow$ (iv) By Lemma 1.2.10(i)

$A^TA = \rho' J + (k-\rho')I - (\rho'-\rho)T$, where T is an adjacency matrix of $G(\underline{B},\rho)$. From Lemma 2.5.2 T has eigenvalues $\theta_0 = \{(k-\rho') - (rk-b\rho')\}/(\rho'-\rho)$; $\theta_1 = (k-\rho')/(\rho'-\rho)$ and $\theta_2 = \{(k-\rho') - (r-\lambda)\}/(\rho'-\rho)$. But $\rho' = k^2/v$ and so $\theta_0 = \theta_1$, i.e. $G(\underline{B},\rho) \cong \Gamma(c,m)$ by Result 1.2.8.

(iv) $\Rightarrow$ (v) Trivial.

(v) $\Rightarrow$ (vi) Label the points of $\underline{D}$: $P_1,\ldots,P_v$, and label the block classes of the group division of $\underline{D}^*$: $\underline{B}_1,\ldots,\underline{B}_c$. Then suppose that P_i is incident with γ_{ij} blocks of $\underline{B}_j$. Lemma 2.1.7 gives :-

$$\lambda k = \rho' \sum_{w=1}^{c} \gamma_{iw} + (\rho-\rho')\gamma_{ij} + (k-r+\lambda-\rho)\delta \quad \text{for every } i,j$$

$(1\leq i\leq v; 1\leq j\leq c)$. By Result 2.5.1 $\rho = k-r+\lambda$, and we have :

$$\lambda k = \rho' r + (\rho-\rho')\gamma_{ij} \quad \text{(using Result 2.4.1)}$$

i.e. $\gamma_{ij} = (\rho' r-\lambda k)/(\rho'-\rho)$, and so every point is incident with the same number of blocks of each block class, and so, by Result 1.4.5, $\underline{D}^*$ is SRGD.

(vi) ⇒ (vii) Trivial.

(vii) ⇒ (viii) Let $\underline{P}_1,\ldots,\underline{P}_d$ be the SRP Division of $\underline{D}^*$. By Theorem 2.2.1(i), $\underline{P}_1,\ldots,\underline{P}_d$ forms a resolution of $\underline{D}$, which is strong by Theorem 2.4.4 and hence $b+d = v+1$.

(viii) ⇒ (i) The P Division is a CLP Division by Result 2.5.1 and (i) follows from Theorem 2.5.7. ¤

Remark 1. Thus all quasi-symmetric 2-designs with block graph $\Gamma(c,m)$ have been characterised. Goethals and Seidel in [20] obtain similar results for quasi-symmetric designs with block graphs the ladder graph and the complement of the ladder graph. John [25] and Goethals and Seidel [20] have also shown that the block graph of a quasi-symmetric design can never be the lattice graph $L_2(n)$ or its complement.

Remark 2. Theorems 2.5.6, 2.5.7 and 2.5.9 provide necessary and sufficient conditions for a design to be strongly resolvable. Further results of this type are Result 1.5.3 and theorems of Shrikhande and Raghavarao, [44] and Shah, [39].