Constructing orientable and negative orientable sequences with asymptotically optimal period

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Abstract

Orientable sequences, periodic sequences in which any n-tuple appears at most once in either direction, were introduced in the early 1990s for use in certain position location applications; constructions and upper bounds on the period for the binary case were published by Dai et al. More recent work has focussed on k-ary sequences for arbitrary $k > 2$; one method of construction involves negative orientable sequences, in which an n-tuple appears at most once in either the sequence or the negative of its reverse. In this paper we show how additional n-tuples can be added to one previously described approach for generating negative orientable sequences, resulting in new sequences with asymptotically optimal period. These sequences can in turn be used to generate orientable sequences, again with asymptotically optimal period.

1 Introduction

1.1 Background

Orientable sequences were introduced in the early 1990s [3, 4, 6] in the context of a position sensing application. An orientable sequence of order n is an infinite periodic sequence with symbols drawn from a finite alphabet — typically $\mathbb{Z}_k$ for some k — with the property that any particular subsequence of length n, referred to throughout as an n-tuple, occurs at most once in a period *in either direction*. That is, if anyone reading the sequence observes n consecutive symbols, they can deduce both the direction in which they are reading and their position within one period of the sequence. Gabrić and Sawada [9] provide an interesting discussion of further possible applications as well as their relationship to strings relevant to DNA computing. Note that for the purposes of this article the *period* of a sequence (a_i) is the least positive integer n such that $a_{i+n} = a_i$ for every i.

The early work referred to above focussed on the binary case, i.e. where $k = 2$; Dai et al. [6] provided both an upper bound on the period for this case and a method of construction yielding sequences with asymptotically optimal periods. More recently, Mitchell and Wild [11] showed how the Lempel homomorphism [10] could be applied to recursively generate binary orientable sequences with periods a large fraction of the optimal value. In 2024, Gabrić and Sawada [7, 8] described a highly efficient method of generating binary orientable sequences with the largest known periods.

In 2024, Alhakim et al. [2] studied the general alphabet case, i.e. where $k > 2$. They gave an upper bound on the period of orientable sequences for all n and k, and also described a range of methods of construction using the Alhakim and Akinwande generalisation of the Lempel homomorphism to arbitrary finite alphabets [1]. Since then a range of construction methods have been proposed [9, 13, 12]; of particular interest is the method of Gabrić and Sawada [9], who showed how to construct sequences with asymptotically optimal period for any n and $k > 2$ using a cycle-joining approach. The construction method described in this paper is thus not the first to yield sequences with asymptotically optimal period, but for the values of n and k we investigated the periods of the generated sequences are greater in every case.

1.2 Motivation and structure

The notion of the *pseudoweight* of a k-ary n-tuple is central to this paper. The pseudoweight of an n-tuple is simply the integer sum of the elements in the tuple, added to $k/2$ times the number of zeros, i.e. the weight of the tuple if every 0 is replaced by $k/2$. The pseudoweight of any n-tuple clearly lies between n and $n(k-1)$.

A second concept fundamental here is the set $E_k(n-1)$ of k-ary n-tuples with pseudoweight less than the 'middle' value $nk/2$. There are $\frac{k^n - r_{k,n,kn/2}}{2}$ such tuples, where $r_{k,n,s}$ is the number of k-ary n-tuples with pseudoweight s. In an earlier paper [13] we showed how to construct a family of *negative orientable sequences*, where a negative orientable sequence is one in which an n-tuple occurs only once in the period of a sequence or the negative of its reverse, where the negative is computed modulo k. The period of such a sequence is $|E_k(n-1)| = \frac{k^n - r_{k,n,kn/2}}{2}$, which approaches the maximum possible; this sequence includes precisely the tuples in $E_k(n-1)$. This in turn enables us to construct orientable sequences with large period. The key properties making the construction possible are that: (a) $E_k(n-1)$ possesses the antinegasymmetry property, i.e. the reverse of the negative of any n-tuple in $E_k(n-1)$ is not in $E_k(n-1)$ — this is simple to see since the negative of a n-tuple with pseudoweight s has pseudoweight $nk/2 - s$, and

(b) $E_k(n-1)$ is Eulerian i.e. it is the set of edges of an Eulerian subgraph of the de Bruijn digraph $B_k(n-1)$ (defined in Subsection 1.3 below).

The goal of this paper is to consider how we might add certain n-tuples of pseudoweight $kn/2$ to $E_k(n-1)$ whilst preserving its antinegasymmetry and its Eulerian property. This enables us to construct negative orientable sequences of period greater than $\frac{k^n - r_{k,n,kn/2}}{2}$, and indeed of asymptotically optimal period.

The remainder of the paper is organised as follows. Following this introductory section, in Section 2 an approach to dividing all the n-tuples of pseudoweight exactly $nk/2$ into circuits is described; this partition is of fundamental importance to the rest of the paper. Section 3 then describes how the n-tuples in certain of these circuits can be added to the set $E_k(n-1)$ of k-ary n-tuples with pseudoweight less than $nk/2$ without losing the key properties enabling the construction of a negative orientable sequence. This leads to Section 4, in which a lower bound is developed on the number of n-tuples that can be added to $E_k(n-1)$. The final main part of the paper, Section 5, shows how new orientable and negative orientable sequences can be constructed from the enlarged version of $E_k(n-1)$, and lower bounds on the sequence lengths are obtained; it is also shown that the sequences have asymptotically optimal period. The paper concludes in Section 6 with a brief discussion of possible future work.

1.3 Notation and basic results

We next formalise some of the key concepts we introduced in the introductory text. In this paper we consider periodic sequences (s_i) with elements from $\mathbb{Z}_k$ for some k, which we refer to as k-ary. Since we are interested in tuples occurring either forwards or backwards in a sequence, we also introduce the notion of a reversed tuple, so that if $\mathbf{u} = (u_0, u_1, \ldots, u_{n-1})$ is a k-ary n-tuple then $\mathbf{u}^R = (u_{n-1}, u_{n-2}, \ldots, u_0)$ is its *reverse*. We are also interested in negating all the elements of a tuple, and hence if $\mathbf{u} = (u_0, u_1, \ldots, u_{n-1})$ is a k-ary n-tuple, we write $-\mathbf{u}$ for $(-u_0, -u_1, \ldots, -u_{n-1})$. We write $\mathbf{s}_n(i)$ for the tuple $(s_i, s_{i+1}, \ldots, s_{i+n-1})$.

Definition 1.1 ([2]). A k-ary *n-window sequence* $S = (s_i)$ (also known as a *cut-down de Bruijn sequence* — see, for example, [5]) is a periodic sequence of elements from $\mathbb{Z}_k$ ($k > 1$, $n > 1$) with the property that no n-tuple appears more than once in a period of the sequence, i.e. with the property that if $\mathbf{s}_n(i) = \mathbf{s}_n(j)$ for some i, j, then $i \equiv j \pmod{m}$ where m is the period of the sequence.

This paper is concerned with two special classes of n-window sequences: orientable and negative orientable sequences.

Definition 1.2 ([2])**.** A k-ary n-window sequence $S = (s_i)$ is said to be an *orientable sequence of order* n, an $\mathcal{OS}_k(n)$, if $\mathbf{s}_n(i) \neq \mathbf{s}_n(j)^R$, for any i, j.

Definition 1.3 ([2])**.** A k-ary n-window sequence $S = (s_i)$ is said to be a *negative orientable sequence of order* n, a $\mathcal{NOS}_k(n)$, if $\mathbf{s}_n(i) \neq -\mathbf{s}_n(j)^R$, for any i, j.

As described above, the notion of the pseudoweight of a k-ary n-tuple, as defined in [12], is central to this note — we write w^* for the pseudoweight function. Using the notation of [12], we also write $r_{k,n,s}$ for the number of k-ary n-tuples with pseudoweight exactly s, where $n \leq s \leq n(k-1)$.

Let $B_k(n-1)$ be the de Bruijn digraph with vertices labeled with k-ary $(n-1)$-tuples and edges labeled with k-ary n-tuples. Following [13], let $E_k(n-1)$ be the set of edges in $B_k(n-1)$ with pseudoweight less than $kn/2$. Then, from [13, Theorem 5.3], the subgraph of $B_k(n-1)$ with edge set $E_k(n-1)$ and vertices those which have an outgoing or incoming edge in $E_k(n-1)$ contains $\frac{k^n - r_{k,n,kn/2}}{2}$ edges and every vertex has in-degree equal to its out-degree; moreover, by the argument in [12, Section 3.3] this subgraph is connected, and hence it is Eulerian.

A k-ary n-tuple $\mathbf{a}$ is said to be negasymmetric if $\mathbf{a} = -\mathbf{a}^R$. We extend this terminology to $B_k(n-1)$ and refer to an edge or vertex being negasymmetric if its associated n-tuple or $(n-1)$-tuple respectively is negasymmetric. A negasymmetric edge must have pseudoweight $kn/2$. Such an edge cannot belong to an $\mathcal{NOS}_k(n)$. We are interested in those edges of pseudoweight $kn/2$ which are not negasymmetric and the possibility that a proportion of them might be joined to $E_k(n-1)$ that preserves the Eulerian property and hence can be used to yield an $\mathcal{NOS}_k(n)$ of larger period. The following elementary results will be of use later.

Lemma 1.1. *Suppose $k \geq 3$ and $n \geq 3$. The number of negasymmetric n-tuples is*

$$\begin{array}{rl} 2k^{\lfloor n/2 \rfloor} & \textit{if } k \textit{ is even and } n \textit{ is odd, and} \\ k^{\lfloor n/2 \rfloor} & \textit{otherwise.} \end{array}$$

Proof. The first $\lfloor n/2 \rfloor$ elements of a negasymmetric n-tuple determine the last $\lfloor n/2 \rfloor$ elements. Moreover, if n is odd then the middle element of the tuple must be 0 or $k/2$. The result follows. $\square$

Lemma 1.2. *Suppose $(a_0, a_1, \ldots, a_{n-1})$ is a negasymmetric n-tuple. Then the following n-tuples are also negasymmetric:*

$$\begin{array}{rl} (a_{n/2}, a_{n/2+1}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{n/2-1}) & \textit{if } n \textit{ is even, and} \\ (a_{(n+1)/2}, a_{(n+1)/2+1}, \ldots, a_{n-1}, x, a_0, a_1, \ldots, a_{(n-3)/2}) & \textit{if } n \textit{ is odd.} \end{array}$$

where x is either 0 or $k/2$.

Proof. The result follows immediately from the definition of negasymmetric. $\square$

Again following [13], a subgraph of $B_k(n-1)$ is said to be antinegasymmetric if $\mathbf{a} \neq -\mathbf{b}^R$ for every pair of edges, $(\mathbf{a}, \mathbf{b})$, in this subgraph. The subgraph of $B_k(n-1)$ with edge set $E_k(n-1)$ is clearly antinegasymmetric, and since it is also Eulerian there exists a Eulerian circuit in this subgraph corresponding to a sequence S which is an $\mathcal{NOS}_k(n)$ of period $\frac{k^n - r_{k,n,kn/2}}{2}$ (see [12, Lemma 3.15]). The reverse of $-S$ covers the edges of pseudoweight greater than $kn/2$ (as, of course, does $-S$).

Let $H_k(n-1)$ be the subgraph of $B_k(n-1)$ restricted to edges of pseudoweight precisely $kn/2$. The nature of $H_k(n-1)$ will be quite different depending on whether n is odd or even and hence we consider the cases separately. Observe that, by definition, there are $r_{k,n,kn/2}$ edges in $H_k(n-1)$.

2 Partitioning $H_k(n-1)$ into necklaces

We first need the following concept that is key to the ideas in this paper.

Definition 2.1 (Necklace — see, for example, [14]). A k-ary *necklace* is a cycle of period dividing n within the de Bruijn graph $B_k(n-1)$ that includes all edges (n-tuples) that are cyclic shifts of one another. If $(a_0, a_1, \ldots, a_{n-1})$ is a k-ary n-tuple, i.e. an edge in $B_k(n-1)$, then let m be the smallest positive integer c such that $a_i = a_{\overline{i+c}}$ for every i ($0 \leq i < n$), where $\overline{x} = x \bmod n$. We write $[a_0, a_1, \ldots, a_{n-1}]$ for the necklace in $B_k(n-1)$ consisting of edges

$$(a_0, a_1, \ldots, a_{n-1}), (a_1, a_2 \ldots, a_{n-1}, a_0), \ldots, (a_{m-1}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{m-2}),$$

and say the necklace has period m.

We start by showing how the edges of $H_k(n-1)$ can readily be divided into edge-disjoint necklaces.

2.1 Defining the partition

We first establish the following.

Lemma 2.1. *Suppose $k \geq 3$ and $n \geq 3$. Suppose $\mathbf{a} = (a_1, a_2 \ldots, a_{n-1})$ is a vertex of $H_k(n-1)$. Then:*

- *if k is odd then $\mathbf{a}$ has in-degree and out-degree:*

$$\begin{array}{ll} 1 & \text{if } \; w^*(\mathbf{a}) \in \{kn/2 - i : 1 \leq i \leq k-1\} \text{ or } w^*(\mathbf{a}) = k(n-1)/2 \\ 0 & \text{otherwise.} \end{array}$$

- *if k is even then* $\mathbf{a}$ *has in-degree and out-degree:*

$$\begin{array}{ll} 2 & \textit{if}\ \ w^*(\mathbf{a}) = k(n-1)/2 \\ 1 & \textit{if}\ \ w^*(\mathbf{a}) \in \{kn/2 - i : 1 \le i \le k-1\} \\ 0 & \textit{otherwise.} \end{array}$$

Proof. If $(a_0, a_1, a_2 \ldots, a_{n-1})$ is an edge incoming to $\mathbf{a}$ then, since this edge must have pseudoweight $kn/2$, it follows by definition that

$$\sum_{i=0}^{n-1} w^*(a_i) = kn/2,$$

i.e.

$$\sum_{i=1}^{n-1} w^*(a_i) = kn/2 - w^*(a_0).$$

- Suppose k is odd. Since $w^*(a_0) \in \{i : 1 \le i \le k-1\}$ or $w^*(a_0) = k/2$, the result follows given that $w^*(a_0) = k/2$ if and only if $a_0 = 0$.
- Suppose k is even. Exactly the same argument applies except that $w^*(a_0) = k/2$ if and only if $a_0 = 0$ or $a_0 = k/2$.

□

We next present a simple way of dividing the edges (n-tuples) in $H_k(n-1)$ into edge-disjoint necklaces.

Lemma 2.2. *Suppose $k \ge 3$ and $n \ge 3$.*

i) If $(a_0, a_1, \ldots, a_{n-1})$ is a k-ary n-tuple of pseudoweight $kn/2$, i.e. an edge in $H_k(n-1)$, then $[a_0, a_1, \ldots, a_{n-1}]$ is a necklace in $H_k(n-1)$ of period dividing n.

ii) Let $\mathcal{C}_k(n-1)$ be the set of all necklaces of the form $[a_0, a_1, \ldots, a_{n-1}]$, where $(a_0, a_1, \ldots, a_{n-1})$ is an edge in $H_k(n-1)$; then $\mathcal{C}_k(n-1)$ forms an edge-disjoint partition of the edges in $H_k(n-1)$.

Proof. i) Suppose $(a_0, a_1, \ldots, a_{n-1})$ is any edge in $H_k(n-1)$. Then every cyclic shift of $(a_0, a_1, \ldots, a_{n-1})$ is also an edge in $H_k(n-1)$, since cyclic shifting does not alter pseudoweight. That is, $[a_0, a_1, \ldots, a_{n-1}]$ is a necklace in $H_k(n-1)$ with period that divides n.

ii) Any edge $(a_0, a_1, \ldots, a_{n-1})$ is in a unique necklace of this type, and hence these necklaces cover all possible edges and are edge-disjoint.

□

Corollary 2.3. *If $k \geq 3$ is odd and $n \geq 3$ then the necklaces in $\mathcal{C}_k(n-1)$ form a unique vertex- and edge-disjoint partition of $H_k(n-1)$.*

Proof. From Lemma 2.1, if k is odd, every vertex in $H_k(n-1)$ has in-degree and out-degree at most one, and hence the necklaces in $\mathcal{C}_k(n-1)$ are vertex-disjoint. Finally, the fact that this partition is unique follows immediately from the fact that the in-degree (and out-degree) of every vertex in $H_k(n-1)$ is at most one. □

Remark 2.1. Whilst the partition $\mathcal{C}_k(n-1)$ is a unique partition of the edges in $H_k(n-1)$ when k is odd, this is not the case for k even; neither is the partition vertex-disjoint in the k even case. That is, there may be other partitions for the k even case which give better results than those we achieve below using $\mathcal{C}_k(n-1)$. This remains an area for future work.

The above lemma motivates the following extension of the notion of negasymmetry.

Definition 2.2. A necklace in $H_k(n-1)$ is said to be *negasymmetric* if there are edges, $\mathbf{a}$ and $\mathbf{b}$ say (not necessarily distinct), in this necklace such that $\mathbf{a} = -\mathbf{b}^R$. A necklace is *not* negasymmetric, i.e. *non-negasymmetric*, only if the subgraph of $B_k(n-1)$ it defines is antinegasymmetric.

Of course, if the necklace contains a negasymmetric edge then clearly the circuit is negasymmetric.

We have the following simple technical results.

Lemma 2.4. *Suppose $n \geq 3$, $k \geq 3$, and $c \mid n$. Then:*

(i) Suppose $(a_0, a_1, \ldots, a_{n-1})$ is a k-ary n-tuple with $a_i = a_{i+c}$ for every i $(0 \leq i \leq n-c-1)$. Then $(a_0, a_1, \ldots, a_{n-1})$ is a negasymmetric n-tuple if and only if $(a_0, a_1, \ldots, a_{c-1})$ is a negasymmetric c-tuple. Moreover, in this case if c is odd then $a_{(c-1)/2} = 0$ or $k/2$ (the latter only applying if k is even).

(ii) Suppose $[a_0, a_1, \ldots, a_{n-1}]$ is a necklace in $B_k(n-1)$ of period dividing c. Then $[a_0, a_1, \ldots, a_{n-1}]$ is a negasymmetric necklace in $B_k(n-1)$ if and only if $[a_0, a_1, \ldots, a_{c-1}]$ is a negasymmetric necklace in $B_k(c-1)$. Moreover, $[a_0, a_1, \ldots, a_{n-1}]$ is a negasymmetric necklace in $H_k(n-1)$ if and only if $[a_0, a_1, \ldots, a_{c-1}]$ is a negasymmetric necklace in $H_k(c-1)$.

Proof. (i) If $(a_0, a_1, \ldots, a_{n-1})$ is negasymmetric, then $a_i = -a_{n-1-i}$ for every i $(0 \leq i \leq n-1)$. But, since $a_i = a_{i+c}$ for every i and $c \mid n$, $a_{n-1-i} = a_{c-1-i}$ for every i, and hence $(a_0, a_1, \ldots, a_{c-1})$ is negasymmetric. If $(a_0, a_1, \ldots, a_{c-1})$ is negasymmetric, then $a_i = -a_{c-1-i}$ for every i $(0 \leq i \leq c-1)$. But $a_{tc+i} = a_i$ for every t $(0 \leq t \leq n/c-1)$, and hence $a_i = -a_{n-1-i}$ for every i, and so $(a_0, a_1, \ldots, a_{n-1})$ is negasymmetric. Finally, if $(a_0, a_1, \ldots, a_{c-1})$ is negasymmetric and c is odd then, by definition, $a_{(c-1)/2} = -a_{(c-1)/2}$ and the result follows.

(ii) If $[a_0, a_1, \ldots, a_{n-1}]$ is negasymmetric then $a_i = -a_{\overline{t-i}}$ for some t and every i $(0 \leq i \leq n-1)$, where the bar indicates reduction modulo n. Hence $a_i = -a_{\overline{\overline{t-i}}}$, where the double bar indicates reduction modulo c, since $c \mid n$. Hence $[a_0, a_1, \ldots, a_{c-1}]$ is negasymmetric. If $[a_0, a_1 \ldots, a_{c-1}]$ is negasymmetric, then $a_i = -a_{\overline{\overline{t-i}}}$ for some t and every i $(0 \leq i \leq c-1)$. But $a_{tc+i} = a_i$ for every t $(0 \leq t \leq n/c-1)$, and hence $a_i = -a_{\overline{t-i}}$ for every i $(0 \leq i \leq n-1)$, and thus $[a_0, a_1 \ldots, a_{n-1}]$ is negasymmetric. Finally, it is immediate that $[a_0, a_1, \ldots, a_{n-1}]$ has pseudoweight $kn/2$ if and only if $[a_0, a_1, \ldots, a_{c-1}]$ has pseudoweight $kc/2$.

□

Lemma 2.5. *Suppose $n \geq 3$, $k \geq 3$ and $[a_0, a_1, \ldots, a_{n-1}]$ is a necklace in $B_k(n-1)$ containing two negasymmetric n-tuples at distance t apart; without loss of generality suppose $0 < t \leq n/2$. Then $[a_0, a_1 \ldots, a_{n-1}]$ has period dividing $2t$.*

Proof. Without loss of generality suppose the two negasymmetric n-tuples are

$$(a_0, a_1, \ldots, a_{n-1}) \quad \text{and} \quad (a_{\overline{t}}, a_{\overline{t+1}} \ldots, a_{\overline{t-1}})$$

where $\overline{s} = s \bmod n$. Then, by definition

$$a_{\overline{i}} = -a_{\overline{-1-i}} \quad \text{and} \quad a_{\overline{t+i}} = -a_{\overline{t-1-i}}$$

for every i $(0 \leq i < n)$. Setting $j = i + t$ in the second equation gives

$$a_{\overline{j}} = -a_{\overline{2t-1-j}}$$

for every j $(0 \leq j < n)$. Hence

$$a_{\overline{-1-i}} = a_{\overline{2t-1-i}}$$

for every i $(0 \leq i < n)$, and hence

$$a_{\overline{i}} = a_{\overline{2t+i}}$$

for every i $(0 \leq i < n)$. The desired result follows. □

Corollary 2.6. *Suppose $n \geq 3$, $k \geq 3$, and that $[a_0, a_1, \ldots, a_{n-1}]$ has period c and contains two distinct negasymmetric n-tuples. Then c is even and the two negasymmetric n-tuples must be at distance exactly $c/2$ apart.*

Proof. Suppose the two negasymmetric n-tuples are at distance t apart, where, without loss of generality, $0 < t \leq c/2$. Since the two tuples are distinct, c cannot be a factor of t; however, by Lemma 2.5, $c \mid 2t$, and hence c must be even. Since $c \mid 2t$ and $t \leq c/2$ we have $t = c/2$, and the result follows. $\square$

2.2 Examples

To illustrate the above results we next give some simple examples.

Example 2.1. Suppose $n = k = 3$. First observe that, from [12, Table 2], $r_{k,n,kn/2} = r_{3,3,4.5} = 7$. The seven edges in $H_3(2)$ can be partitioned into three necklaces:

$$[000], [012], [021]$$

of periods 1, 3 and 3, respectively. All three necklaces are negasymmetric and contain one negasymmetric 3-tuple, accounting for the three negasymmetric 3-tuples (see Lemma 1.1).

Example 2.2. Suppose $n = 3$ and $k = 5$. From [12, Table 2], $r_{k,n,kn/2} = r_{5,3,7.5} = 13$. The 13 edges in $H_5(2)$ can be partitioned into five necklaces:

$$[000], [014], [041], [023], [032]$$

all of which have period 3 except the first. All five necklaces are negasymmetric and contain one negasymmetric 3-tuple, accounting for the five negasymmetric 3-tuples (see Lemma 1.1).

Example 2.3. Suppose $n = 4$ and $k = 3$. From [12, Table 2], $r_{k,n,kn/2} = r_{3,4,6} = 19$. The 19 edges in $H_3(3)$ can be partitioned into six necklaces:

$$[0000], [1212], [0012], [0021], [0102], [1122]$$

all of which have period 4 except the first and second (with periods 1 and 2 respectively). All six necklaces are negasymmetric. [0102] contains no negasymmetric 4-tuples, [0000] contains one negasymmetric 4-tuple, and the other four necklaces contain two negasymmetric 4-tuples. We write $N_0 = 1, N_1 = 1, N_2 = 4$ to denote this. This accounts for the nine negasymmetric 4-tuples (see Lemma 1.1).

Example 2.4. Suppose $n = 5$ and $k = 3$. From [12, Table 2], $r_{k,n,kn/2} = r_{3,5,7.5} = 51$. The 51 edges in $H_3(4)$ can be partitioned into 11 necklaces, all but the first of which have period 5:

$$[00000], [00012], [00021], [00102], [00201], 01122], [01212],$$
$$[02211], [02121], [01221], [02112].$$

The first nine of these necklaces are negasymmetric and contain one negasymmetric 3-tuple, but the last two, i.e. [01221] and [02112], are not negasymmetric. That is, as we discuss below, one of these two necklaces could be joined to S to result in a $\mathcal{NOS}_3(5)$ of period five greater than the period given in [12, Table 3], i.e. 96+5=101. The set of 101 5-tuples making up S can, in turn, be used to construct an $\mathcal{OS}_3(6)$ of period $k \times 101 = 303$ (greater than the previous record of 288 and close to the best known upper bound of 315).

Example 2.5. Suppose $n = 6$ and $k = 3$. From [12, Table 2], $r_{k,n,kn/2} = r_{3,6,9} = 141$. The 141 edges in $H_3(5)$ can be partitioned into 26 necklaces: one of period 1, one of period 2, two of period 3, and 22 of period 6. Eight of these 26 necklaces, all of period 6, are non-negasymmetric, namely:

$$[001221], [002112], [010122], [020211], [010212], [020121], [010221], [020112].$$

A further three necklaces of period 6 are negasymmetric, but do not contain any negasymmetric 6-tuples, namely:

$$[000102], [000201], [011022].$$

Three of the remaining necklaces, namely those with periods 1 and 3, are negasymmetric and contain a single negasymmetric 6-tuple:

$$[000000], \quad [012012], [021021].$$

Observe that the two necklaces of period 3 yield the same sequences as the negasymmetric necklaces [012] and [021] in $B_3(2)$. Finally, the remaining 12 necklaces, one of period 2 and eleven of period 6, contain two negasymmetric 6-tuples:

$$[121212], [000012], [000021], [001002], [001122], 002211],$$
$$[001212], [002121], [012021], [111222], [112212], [221121].$$

Using the previous notation, this means that $N_0 = 3$, $N_1 = 3$ and $N_2 = 12$; this accounts for the 27 negasymmetric 6-tuples (see Lemma 1.1).

So, as discussed below, the 6-tuples in four of the eight non-negasymmetric 6-tuples, i.e. a total of 24 6-tuples, could be joined to S to result in a $\mathcal{NOS}_3(6)$ of period 24 greater than the period given in [12, Table 3], i.e. 294+24=318. The set of 318 6-tuples making up S can, in turn, be used to construct an $\mathcal{OS}_3(7)$ of period $k \times 318 = 954$ (greater than the previous record of 882 and close to the best known upper bound of 972).

Example 2.6. Suppose $n = 3$ and $k = 4$. From [12, Table 2], $r_{k,n,kn/2} = r_{4,3,6} = 20$. The 20 edges in $H_4(2)$ can be partitioned into 8 necklaces: two of period 1 and six of period 3, all of which are negasymmetric. All the eight necklaces contain a single negasymmetric 3-tuple, accounting for the eight negasymmetric 3-tuples (see Lemma 1.1).

Example 2.7. Suppose $n = 4$ and $k = 4$. From [12, Table 2], $r_{k,n,kn/2} = r_{4,4,8} = 70$. The 70 edges in $H_4(3)$ can be partitioned into 20 necklaces: two of period 1, two of period 2 and 16 of period 4. Four of the necklaces, all of period 4, are non-negasymmetric, namely:

$$[0132], [0213], [0231], [0312].$$

A further seven necklaces, six of of period 4 and one of period 2 are negasymmetric, but do not contain any negasymmetric 4-tuples, namely:

$$[0002], [0222], [0103], [0123], [0321], [1232], \quad [0202].$$

Two of the remaining necklaces, namely those with period 1, are negasymmetric and contain a single negasymmetric 4-tuple:

$$[0000], [2222].$$

Finally, the remaining seven necklaces, one of period 2 and six of period 4, each contain two negasymmetric 4-tuples:

$$[1313], \quad [0022], [0013], [1003], [2213], [2231], [1133].$$

This means that $N_0 = 7$, $N_1 = 2$ and $N_2 = 7$, accounting for the 16 negasymmetric 4-tuples (see Lemma 1.1).

So, the 4-tuples in two of the four non-negasymmetric 4-tuples, i.e. a total of eight 4-tuples, could be joined to S to result in a $\mathcal{NOS}_4(4)$ of period 8 greater than the period given in [12, Table 3], i.e. 93+8=101. The set of 101 4-tuples making up S can, in turn, be used to construct an $\mathcal{OS}_4(5)$ of period $k \times 101 = 404$.

In the remainder of the paper we generalise the observations made in the examples to obtain formulae for the number of n-tuples that can be added to $E_k(n-1)$ whilst preserving the key properties that enable us to construct a negative orientable sequence containing precisely these tuples.

2.3 Properties of necklaces in $\mathcal{C}_k(n-1)$

As mentioned previously, the goal of this paper is to consider how we might add certain n-tuples of pseudoweight $kn/2$ to $E_k(n-1)$ whilst preserving its antinegasymmetry and its Eulerian property. This would enable us to construct negative orientable sequences of period greater than $\frac{k^n - r_{k,n,kn/2}}{2}$.

To achieve this goal we need to consider the properties of the necklaces in $\mathcal{C}_k(n-1)$. We first have the following key result for odd k.

Lemma 2.7. *Suppose $k \geq 3$ is odd and $n \geq 3$. Suppose X is an Eulerian and antinegasymmetric subset of the edges of $B_k(n-1)$ containing $E_k(n-1)$ as a subset. Then $X - E_k(n-1)$ only contains edges corresponding to complete necklaces from $\mathcal{C}_k(n-1)$, and these necklaces cannot be negasymmetric.*

Proof. First observe that X cannot contain any edges of pseudoweight greater than $kn/2$, and hence $X - E_k(n-1) \subseteq H_k(n-1)$. Since both X and $E_k(n-1)$ are Eulerian, this means that every vertex of the subgraph induced by $X - E_k(n-1)$ has in-degree equal to its out-degree. Since no vertex in $H_k(n-1)$ has in-degree or out-degree greater than one, the fact that $X - E_k(n-1)$ is made up of necklaces follows from Lemma 2.2. Finally, no necklace in $X - E_k(n-1)$ can be negasymmetric, since that would contradict the requirement for X to be antinegasymmetric. □

Remark 2.2. It seems unlikely that the above result is true for k even.

In fact we can say more about the set of necklaces making up $\mathcal{C}_k(n-1)$, regardless of whether k and n are odd or even.

Lemma 2.8. *Suppose $k \geq 3$ and $n \geq 3$. If $[a_0, a_1, \ldots, a_{n-1}]$ is a necklace in $\mathcal{C}_k(n-1)$ then $[-a_0, -a_1, \ldots, -a_{n-1}]$ is also a necklace in $\mathcal{C}_k(n-1)$. Moreover, if $[a_0, a_1, \ldots, a_{n-1}]$ is non-negasymmetric then the reverse-complementary necklace $[-a_{n-1}, -a_{n-2}, \ldots, -a_0]$ is also non-negasymmetric and $[a_0, a_1, \ldots, a_{n-1}]$ shares no edges with $[-a_{n-1}, -a_{n-2}, \ldots, -a_0]$.*

Proof. Since all the edges in $[a_0, a_1, \ldots, a_{n-1}]$ have pseudoweight $nk/2$ then so do all the edges in $[-a_0, -a_1, \ldots, -a_{n-1}]$. Hence $[-a_0, -a_1, \ldots, -a_{n-1}]$ is a necklace in $\mathcal{C}_k(n-1)$.

Next suppose $[a_0, a_1, \ldots, a_{n-1}]$ is non-negasymmetric and $[a_0, a_1, \ldots, a_{n-1}] = [-a_{n-1}, -a_{n-2}, \ldots, -a_0]$, where equality here means that they are the same necklace. Hence $(a_0, a_1, \ldots, a_{n-1}) = (-a_{\overline{t}}, -a_{\overline{t-1}}, \ldots, -a_{\overline{t+1}})$ for some t, where here, as throughout, $\bar{u}$ denotes the unique integer satisfying $0 \leq \overline{u} \leq n-1$ and $u \equiv \overline{u} \pmod{n}$. But this means that $[a_0, a_1, \ldots, a_{n-1}]$ is negasymmetric, giving a contradiction.

Finally, if $[-a_{n-1}, -a_{n-2}, \ldots, -a_0]$ is negasymmetric, then

$$(-a_{n-1}, -a_{n-2}, \ldots, -a_0) = (a_{\overline{t}}, a_{\overline{t+1}}, \ldots, a_{\overline{t-1}})$$

for some t, and hence

$$(a_0, a_1, \ldots, a_{n-1}) = (-a_{\overline{t-1}}, -a_{\overline{t-2}}, \ldots, -a_{\overline{t}}),$$

which again contradicts the assumption that the necklace $[a_0, a_1, \ldots, a_{n-1}]$ is non-negasymmetric. □

Remark 2.3. It follows from Lemma 2.8 that the non-negasymmetric necklaces in $\mathcal{C}_k(n-1)$ can be divided into pairs consisting of a necklace and the reverse of its negative. We refer to these as *reverse-complementary pairs*.

More formally, define a mapping $I : H_k(n-1) \to H_k(n-1)$ such that $(a_0, a_1, \ldots, a_{n-1}) \to (-a_{n-1}, -a_{n-2}, \ldots, -a_0)$, i.e. I maps an n-tuple to its reverse-negative. This induces a involution on the set of necklaces $\mathcal{C}_k(n-1)$ where $[a_0, a_1, \ldots, a_{n-1}] \to [-a_{n-1}, -a_{n-2}, \ldots, -a_0]$. Necklaces fixed by I are negasymmetric, and the other orbits under I are of size 2 (making up the reverse-complementary pairs).

Example 2.8. Revisiting Example 2.4, the two non-negasymmetric necklaces [01221] and [02112] form a reverse-complementary pair. Similarly, revisiting Example 2.5, the eight non-negasymmetric necklaces can be divided into four reverse-complementary pairs, as follows:

$$([001221], [002112]); \quad ([010122], [020112]);$$
$$([010212], [020121]); \quad ([010221], [020211]).$$

Similarly, looking at Example 2.7, the four non-negasymmetric necklaces can be divided into two reverse-complementary pairs, as follows:

$$([0132], [0213]), \quad ([0231], [0312]).$$

Because of Lemma 2.7, it is thus of interest to know how many reverse-complementary pairs of non-negasymmetric necklaces there are in $\mathcal{C}_k(n-1)$, and what their periods are. A first step in this direction is the following result.

Lemma 2.9. *Suppose $k \geq 3$ and $n \geq 3$. Suppose $[a_0, a_1, \ldots, a_{n-1}]$ is a negasymmetric necklace in $\mathcal{C}_k(n-1)$ of period c, where c is odd ($c \mid n$). Then $(a_{\overline{s}}, a_{\overline{s+1}}, \ldots, a_{\overline{s-1}})$ is a negasymmetric n-tuple for some s ($0 \leq s \leq n-1$). Moreover, $[a_0, a_1, \ldots, a_{n-1}]$ contains precisely one negasymmetric n-tuple.*

Proof. Since $[a_0, a_1, \ldots, a_{n-1}]$ is negasymmetric, by definition we have

$$(a_0, a_1, \ldots, a_{n-1}) = (-a_{\overline{t}}, -a_{\overline{t-1}}, \ldots, -a_{\overline{t+1}})$$

for some t. Hence

$$a_{\overline{i}} = -a_{\overline{t-i}}$$

for every i, $0 \leq i < n$.

If t is odd, setting $i = (t+1)/2 + j$ gives

$$a_{\overline{(t+1)/2+j}} = -a_{\overline{(t+1)/2-1-j}},$$

for every j, $0 \leq j < n$. Hence

$$(a_{\overline{s}}, a_{\overline{s+1}}, \ldots, a_{\overline{s-1}}) = (-a_{\overline{s-1}}, -a_{\overline{s-2}}, \ldots, -a_{\overline{s}})$$

where $s = (t+1)/2$, and so $(a_{\overline{s}}, a_{\overline{s+1}}, \ldots, a_{\overline{s-1}})$ is negasymmetric.

If t is even, setting $i = (t-c+1)/2 + j$ (which is an integer since c is odd) gives

$$a_{\overline{(t-c+1)/2+j}} = -a_{\overline{t-(t-c+1)/2-j}} = -a_{\overline{(t-c+1)/2+c-1-j}} = -a_{\overline{(t-c+1)/2-1-j}}$$

for every j, $0 \leq j < n$, since $[a_0, a_1, \ldots, a_{n-1}]$ has period c. Thus

$$(a_{\overline{s}}, a_{\overline{s+1}}, \ldots, a_{\overline{s-1}}) = (-a_{\overline{s-1}}, -a_{\overline{s-2}}, \ldots, -a_{\overline{s}})$$

where $s = (t-c+1)/2$, and so $(a_{\overline{s}}, a_{\overline{s+1}}, \ldots, a_{\overline{s-1}})$ is negasymmetric. That is $[a_0, a_1, \ldots, a_{n-1}]$ contains at least one negasymmetric n-tuple.

The fact that the n-tuple is unique follows immediately from Corollary 2.6.
□

Remark 2.4. Since $c \mid n$, Lemma 2.9 immediately implies that if n is odd then all the negasymmetric necklaces in $\mathcal{C}_k(n-1)$ contain a unique negasymmetric n-tuple. However, the above result does not hold if c is even, and as a result we examine the n odd and even cases separately.

To see why the c even case is different, suppose

$$(a_0, a_1, \ldots, a_{n-1}) = (-a_{\overline{t}}, -a_{\overline{t-1}}, \ldots, -a_{\overline{t+1}})$$

for some t. If t is odd then the same argument as in the above proof shows that there is a negasymmetric n-tuple in the necklace $[a_0, a_1, \ldots, a_{n-1}]$. However, if t is even then the argument used above does not work, since c being odd is key. An example in the case $n = 4$ and $k = 3$ involves the necklace $[a_0 = 1, a_1 = 0, a_2 = 2, a_3 = 0]$. It is simple to see that:

$$(a_0, a_1, a_2, a_3) = (-a_2, -a_1, -a_0, -a_3)$$

i.e. the necklace is negasymmetric (in the above terminology $t = 2$). However, none of the four 4-tuples in the necklace are negasymmetric.

3 Adding necklaces from $\mathcal{C}_k(n-1)$ to $E_k(n-1)$

As mentioned previously, the goal of this paper is to investigate how we can add additional n-tuples to the set $E_k(n-1)$ while maintaining the property that the corresponding subgraph of $B_k(n-1)$ is both Eulerian and antinegasymmetric. We can now give our main result.

Theorem 3.1. *Suppose $k \geq 3$ and $n \geq 3$. Suppose $X_k(n-1)$ is the subset of edges of $B_k(n-1)$ made up of the union of $E_k(n-1)$ and the sets of edges from one of each reverse-complementary pair of the non-negasymmetric necklaces in $\mathcal{C}_k(n-1)$. Then:*

(i) the subgraph of $B_k(n-1)$ with edge set $X_k(n-1)$ and vertices with non-zero in-degree is Eulerian;

(ii) the subgraph of $B_k(n-1)$ with edge set $X_k(n-1)$ is antinegasymmetric.

Proof. (i) Since the subgraph of $B_k(n-1)$ with edge set $E_k(n-1)$ is Eulerian (as established in [12, Section 3.3]) and we are adjoining complete necklaces, it is clear that the subgraph of $B_k(n-1)$ with edge set $X_k(n-1)$ has in-degree equal to out-degree for every vertex. It remains to show that it is connected. We know that the subgraph with edge-set $E_k(n-1)$ is connected, and hence it remains to show that every added necklace from $H_k(n-1)$ has a vertex in common with one of the edges in $E_k(n-1)$. Since $E_k(n-1)$ contains all edges of pseudoweight less than $kn/2$ and $k > 2$, all vertices in $B_k(n-1)$ with pseudoweight at most $(kn-3)/2$ have at least one outgoing (and incoming) edge in $E_k(n-1)$ — this follows since if $(v_0, v_1, \ldots, v_{n-2})$ is such a vertex, then $(v_0, v_1, \ldots, v_{n-2}, 1)$ is an outgoing edge in $B_k(n-1)$ of pseudoweight at most $(kn-3)/2 + 1 < kn/2$, i.e. it is an edge in $E_k(n-1)$. Every added necklace contains only edges of pseudoweight $kn/2$, and moreover cannot be the all-zero or all-$kn/2$ necklace, since they are negasymmetric. That is an added necklace must contain at least one element not equal to 1, i.e. such a necklace will pass through a vertex with pseudoweight less than $kn/2 - 1$, i.e. a vertex with an outgoing edge in $E_k(n-1)$.

(ii) $E_k(n-1)$ is negasymmetric, as are all the added necklaces. If $\mathbf{a}$ is an edge in an added necklace it must have pseudoweight $kn/2$, and hence $-\mathbf{a}^R$ must also have pseudoweight $kn/2$; hence $-\mathbf{a}^R$ is not in $E_k(n-1)$. Finally, suppose that $\mathbf{a}$ and $\mathbf{b}$ are edges in distinct added necklaces. Then $-\mathbf{a}^R$ is in the reverse-complementary necklace to the one containing $\mathbf{a}$, and hence $-\mathbf{a}^R \neq \mathbf{b}$.

□

Example 3.1. Reviewing Examples 2.5 and 2.8, in the case $n = 6$ and $k = 3$ the set $X_3(5)$ will contain $E_3(5)$ and four of the eight non-negasymmetric necklaces in $\mathcal{C}_3(5)$, one of each reverse-complementary pair. That is $|X_3(5)| = |E_3(5)| + 24 = 318$, yielding a $\mathcal{NOS}_3(6)$ of period 318.

It therefore remains to determine the size of the set $X_k(n-1)$, as defined in Theorem 3.1, and this is the main focus of the remainder of the paper.

4 Enumerating non-negasymmetric necklaces

4.1 Preliminaries

For the reason given in Remark 2.4, we divide the analysis into two cases, depending whether n is odd or even. However, we first give a key definition and a lemma which apply to all n.

Definition 4.1. Suppose $k \geq 3$ and $n \geq 3$. For $i \geq 0$, let $N_i(n,k)$ represent the set of negasymmetric necklaces in $\mathcal{C}_k(n-1)$ containing i negasymmetric n-tuples.

Remark 4.1. If n is odd, it follows immediately from Lemma 2.9 that $|N_i(n,k)| = 0$ if $i \neq 1$. Also, from Corollary 2.6, if n is even then a negasymmetric necklace $[a_0, a_1, \ldots, a_{n-1}]$ contains zero, one or two negasymmetric n-tuples, i.e. $|N_i(n,k)| = 0$ if $i > 2$.

The following result is elementary.

Lemma 4.1. *Suppose $k \geq 3$ and $n \geq 3$. Let $n = 2^t m$, where $t \geq 0$ and m is odd. Then the number of n-tuples (edges) in $H_k(n-1)$ that are not in a negasymmetric necklace in $\mathcal{C}_k(n-1)$, and hence are in a non-negasymmetric necklace in $\mathcal{C}_k(n-1)$, is at least*

$$r_{k,n,kn/2} - n(|N_0(n,k)| + |N_2(n,k)|) - m(|N_1(n,k)| - \delta) - \delta$$

where $\delta = 1$ or 2 depending on whether k is odd or even, with equality if n is prime.

Proof. Every negasymmetric necklace in $\mathcal{C}_k(n-1)$ contains at most n n-tuples. However, the necklaces in $N_1(n,k)$ have odd period, i.e. period at most m, and hence they contain at most m n-tuples. Moreover, one of the necklaces in $N_1(n,k)$ is the all-zero necklace containing a single n-tuple; if k is even, $N_1(n,k)$ also includes the all-$k/2$ necklace which again contains a single n-tuple. That is, the set of all negasymmetric necklaces contains at most

$$n|N_0(n,k)| + m(|N_1(n,k)| - \delta) + \delta + n|N_2(n,k)|$$

n-tuples. Now $|H_k(n-1)| = r_{k,n,kn/2}$ and the inequality follows.

Finally, suppose n is prime, and hence odd since $n \geq 3$, and so $|N_0(n,k)| = |N_2(n,k)| = 0$. The necklaces in $N_1(n,k)$ must all have 'full' period n, apart from the all-zero and the all-$k/2$ necklaces, and so all but δ of these necklaces contain n distinct n-tuples. Thus in this case equality holds. $\square$

The corollary below follows immediately from Remark 4.1.

Corollary 4.2. *Suppose $k \geq 3$ and $n \geq 3$ is odd. Then the number of n-tuples (edges) in $H_k(n-1)$ that are not in a negasymmetric necklace in $\mathcal{C}_k(n-1)$, and hence are in a non-negasymmetric necklace in $\mathcal{C}_k(n-1)$, is at least*

$$r_{k,n,kn/2} - n(|N_1(n,k)| - \delta) - \delta$$

where $\delta = 1$ or 2 depending on whether k is odd or even, with equality if n is prime.

It thus remains to evaluate $|N_i(n,k)|$.

4.2 The n odd case

We first examine the n odd case, the much simpler of the two. We have the following result.

Lemma 4.3. *Suppose $k \geq 3$ and $n \geq 3$, where n is odd. Then:*

$$|N_1(n,k)| = \delta k^{(n-1)/2}.$$

Proof. Since n is odd, by Lemma 2.9 every negasymmetric necklace in $\mathcal{C}_k(n-1)$ contains a unique negasymmetric n-tuple, and so $|N_1(n,k)|$ is equal to the number of negasymmetric n-tuples, and the result follows from Lemma 1.1. □

4.3 The n even case

As noted above, if $[a_0, a_1, \ldots, a_{n-1}]$ is a negasymmetric necklace of even period in $\mathcal{C}_k(n-1)$ then the claim of Lemma 2.9 does not hold, meaning that it is not so simple to enumerate the negasymmetric necklaces in $\mathcal{C}_k(n-1)$ when n is even. To obtain a result corresponding to Lemma 4.3 we need to analyse further the behaviour of necklaces in $\mathcal{C}_k(n-1)$ when n is even.

We first characterise negasymmetric necklaces in $\mathcal{C}_k(n-1)$ with even period that contain at least one negasymmetric n-tuple. We have the following result, analogous to Lemma 2.9.

Lemma 4.4. *Suppose $k \geq 3$ and $n \geq 4$ is even. Suppose $[a_0, a_1, \ldots, a_{n-1}]$ is a negasymmetric necklace in $\mathcal{C}_k(n-1)$ with even period c containing at least one negasymmetric n-tuple. Then $[a_0, a_1, \ldots, a_{n-1}]$ contains precisely two negasymmetric n-tuples at distance $c/2$ apart.*

Proof. Suppose $(a_{\overline{s}}, a_{\overline{s+1}}, \ldots, a_{\overline{s-1}})$ is negasymmetric, where $\overline{x} = x \bmod n$. Then, if $m = n/2$, it follows immediately that $(a_{\overline{s+m}}, a_{\overline{s+m+1}}, \ldots, a_{\overline{s+m-1}})$ is also negasymmetric, since $\overline{s+m+i} = \overline{s-m+i}$ for every i. The result then follows immediately from Corollary 2.6. □

We also have the following result covering the case where a negasymmetric necklace does not contain any negasymmetric n-tuples.

Lemma 4.5. *Suppose $k \geq 3$, $n \geq 3$, and $[a_0, a_1, \dots, a_{n-1}]$ is a negasymmetric necklace in $\mathcal{C}_k(n-1)$ containing no negasymmetric n-tuples. Then n is even, $[a_0, a_1, \dots, a_{n-1}]$ has period c for some even $c \mid n$, and, for some s $(0 \leq s < c/2)$:*

(i) $a_s = 0$ *or* $k/2$ *and* $a_{s+c/2} = 0$ *or* $k/2$*;*

(ii) *both*

$$(a_{s+1}, a_{s+2}, \dots, a_{c-1}, a_0, a_1, \dots, a_{s-1})$$

and

$$(a_{s+c/2+1}, a_{s+c/2+2}, \dots, a_{c-1}, a_0, a_1, \dots, a_{s+c/2-1})$$

are negasymmetric $(c-1)$-tuples;

(iii) *$[a_0, a_1, \dots, a_{c-1}]$ contains no other negasymmetric $(c-1)$-tuples; moreover, the $(n-1)$-tuples of (ii) are distinct;*

(iv)

$$(a_{s+1}, a_{s+2}, \dots, a_{n-1}, a_0, a_1, \dots, a_{s-1})$$

and

$$(a_{s+c/2+1}, a_{s+c/2+2}, \dots, a_{n-1}, a_0, a_1, \dots, a_{s+c/2-1})$$

are distinct negasymmetric $(n-1)$-tuples, and $[a_0, a_1, \dots, a_{n-1}]$ contains no other negasymmetric $(n-1)$-tuples; also, if $a_s = a_{s+c/2}$

$$(a_{s+n/2+1}, a_{s+n/2+2}, \dots, a_{n-1}, a_0, a_1, \dots, a_{s+n/2-1})$$

equals either

$$(a_{s+1}, a_{s+2}, \dots, a_{n-1}, a_0, a_1, \dots, a_{s-1})$$

or

$$(a_{s+c/2+1}, a_{s+c/2+2}, \dots, a_{n-1}, a_0, a_1, \dots, a_{s+c/2-1})$$

according as n/c is even or odd.

Proof. The fact c is even follows immediately from Lemma 2.9, and since $c \mid n$ it follows that n is also even. By Lemma 2.4, $[a_0, a_1, \dots, a_{c-1}]$ is a negasymmetric necklace in $B_k(c-1)$ which contains no negasymmetric c-tuples. Since $[a_0, a_1, \dots, a_{c-1}]$ is negasymmetric, by definition $a_i = -a_{\overline{t-i}}$ for some t $(0 \leq t \leq c-1)$ and every i $(0 \leq i \leq c-1)$, where the bar indicates computation modulo c.

By Remark 2.4, we know t is even, or else there would exist a negasymmetric c-tuple in the necklace. Hence $a_{t/2} = -a_{t/2}$, i.e. $a_{t/2} = 0$ or $k/2$. If we set

$s = t/2 \bmod c$, then $a_s = a_{t/2} = 0$ or $k/2$, since $[a_0, a_1, \ldots, a_{n-1}]$ has period c. Moreover $0 \leq s < c/2$ by definition of s, establishing part of (i).

It also follows immediately that $a_{s+i} = -a_{\overline{s-i}}$ for every i ($0 \leq i \leq c-1$), where $\overline{x} = x \bmod c$. Thus $(a_{s+1}, a_{s+2}, \ldots, a_{c-1}, a_0, a_1, \ldots, a_{s-1})$ is a negasymmetric $(c-1)$-tuple, establishing the first part of (ii).

But by Lemma 2.4(i), this means that $a_{s+c/2} = 0$ or $k/2$, completing the proof of (i). By Lemma 1.2, it follows that

$$(a_{s+c/2+1}, a_{s+c/2+2}, \ldots, a_{c-1}, a_0, a_1, \ldots, a_{s+c/2-1})$$

is negasymmetric and (ii) follows.

Suppose the first part of (iii) is not true, and without loss of generality relabelling subscripts as necessary, suppose

$$(a_0, a_1, \ldots, a_{c-2}) \text{ and } (a_{\overline{t}}, a_{\overline{t+1}}, \ldots, a_{\overline{t-2}})$$

are both negasymmetric $(c-1)$-tuples where $0 < t < c/2$ and $\overline{x} = x \bmod c$. Thus

$$a_i = -a_{c-2-i} \text{ and } a_{\overline{t+i}} = -a_{\overline{t-2-i}}$$

for every i ($0 \leq i < c-1$).

Also, if $(a_0, a_1, \ldots, a_{c-2})$ is negasymmetric then it must have pseudoweight $(c-1)k/2$, and thus a_{c-1} has pseudoweight $k/2$, and hence $a_{c-1} = 0$ or $k/2$. Similarly we have $a_{t-1} = 0$ or $k/2$. Hence the equations above also hold for $i = c-1$.

Setting $i = t+j$ in the first equation gives $a_{\overline{t+j}} = -a_{\overline{-2-t-j}}$ for every j ($0 \leq j < c$), and thus $a_{\overline{-t-2-i}} = a_{\overline{t-2-i}}$ for every i ($0 \leq i < c$). Hence $[a_0, a_1, \ldots, a_{c-1}]$ has period dividing $2t < c$, which is a contradiction.

The second part of (iii) follows because if the two $(c-1)$-tuples are equal then $a_{\overline{s+i}} = a_{\overline{s+c/2+i}}$ for every i ($1 \leq i \leq c-1$), where the bar denotes reduction modulo c. Hence, setting $i = c/2$, we have $a_{\overline{s+c/2}} = a_{\overline{s+c}} = a_{\overline{s}}$, and so $a_{\overline{s+i}} = a_{\overline{s+c/2+i}}$ for every i ($0 \leq i \leq c-1$). Thus $[a_0, a_1, \ldots, a_{n-1}]$ has period dividing $c/2$, giving the desired contradiction.

Now consider (iv). That these $(n-1)$-tuples are negasymmetric follows immediately, because $(a_0, a_1, \ldots, a_{n-1})$ is a concatenation of n/c copies of $(a_0, a_1, \ldots, a_{c-1})$. Moreover, another negasymmetric $(n-1)$-tuple contained in $[a_0, a_1, \ldots, a_{n-1}]$ would imply another negasymmetric $(c-1)$-tuple contained in $[a_0, a_1, \ldots, a_{c-1}]$. That they are distinct follows from exactly the same argument used to show the $(c-1)$-tuples of (ii) are distinct, except this time the subscripts are computed modulo n.

The statement regarding

$$(a_{s+n/2+1}, a_{s+n/2+2}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{s+n/2-1})$$

is immediate by the periodicity of $[a_0, a_1, \ldots, a_{n-1}]$. $\square$

Remark 4.2. An example of the case where a negasymmetric necklace does not contain any negasymmetric n-tuples for $n = 4$ and $k = 3$ is given in Remark 2.4. Examples of cases where a negasymmetric necklace contains two negasymmetric n-tuples are provided by Examples 2.3 and 2.5.

In addition we need the following simple result covering the case where a negasymmetric necklace $[a_0, a_1, \ldots, a_{n-1}]$ contains precisely one negasymmetric n-tuple

Lemma 4.6. *Suppose $k \geq 3$, $n \geq 4$ is even, and $[a_0, a_1, \ldots, a_{n-1}]$ is a negasymmetric necklace in $\mathcal{C}_k(n-1)$ of odd period. Then $[a_0, a_1, \ldots, a_{n-1}]$ contains precisely one negasymmetric $(n-1)$-tuple.*

Proof. Since $[a_0, a_1, \ldots, a_{n-1}]$ has odd period, m say, then $[a_0, a_1, \ldots, a_{m-1}]$ is a negasymmetric necklace in $\mathcal{C}_k(m-1)$ which must contain a (unique) negasymmetric m-tuple by Lemma 2.9. Suppose this negasymmetric m-tuple is $(d_0, d_1, \ldots, d_{m-1})$. If $(b_0, b_1, \ldots, b_{n-1})$ consists of n/m concatenated copies of $(d_0, d_1, \ldots, d_{m-1})$, then clearly $(b_0, b_1, \ldots, b_{n-1})$ is a negasymmetric n-tuple, and hence $b_i = k - b_{m-1-1}$ for every i, $0 \leq i < n$. It follows immediately that $(b_{(m+1)/2}, b_{(m+3)/2}, \ldots, b_{n-1}, b_0, b_1, \ldots, b_{(m-3)/2})$ is a negasymmetric $(n-1)$-tuple. Hence we need only show that this is unique.

Suppose without loss of generality that $(a_0, a_1, \ldots, a_{n-2})$ and $(a_{\overline{t}}, a_{\overline{t+1}}, \ldots, a_{\overline{t-2}})$ are both negasymmetric $(n-1)$-tuples, for some t $(1 \leq t < m)$, where the bar denotes reduction modulo n. Then

$$a_{\overline{i}} = -a_{\overline{-2-i}}, \quad \text{and} \quad a_{\overline{t+i}} = -a_{\overline{t-2-i}}$$

for every i, $0 \leq i < n-1$. Setting $j = t + i$ in the second equation gives $a_{\overline{j}} = -a_{\overline{2t-2-j}}$ for every j, $0 \leq j < n$ $(j \neq t-1)$. Combining with the first equation this yields $a_{\overline{-2-i}} = -a_{\overline{2t-2-i}}$ for every i, $0 \leq i < n$ $(i \neq n-1$ and $i \neq t-1)$, and setting $j = -2 - i$ means that

$$a_{\overline{j}} = -a_{\overline{2t+j}}$$

for every j, $0 \leq j < n-1$ $(j \neq n-1$ and $j \neq n-1-t)$.

But, observing that $t < m$ and $m \leq n/2$, this means that

$$a_{\overline{j}} = -a_{\overline{2t+j}}$$

for every j, $0 \leq j < m$. Hence $m \mid 2t$, i.e. $m \mid t$ since m is odd. So

$$(a_0, a_1, \ldots, a_{n-2}) = (a_{\overline{t}}, a_{\overline{t+1}}, \ldots, a_{\overline{t-2}})$$

and the desired result follows. $\square$

We also have a partial converse to Lemma 4.5, as follows.

Lemma 4.7. *Suppose $k \geq 3$ and $n \geq 4$. Let $n = 2^t c$, where $t > 0$ and c is odd. If $(a_0, a_1, \ldots, a_{n-2})$ is a negasymmetric $(n-1)$-tuple, then both $[a_0, a_1, \ldots, a_{n-2}, 0]$ and $[a_0, a_1, \ldots, a_{n-2}, k/2]$ (if k is even) are negasymmetric necklaces in $\mathcal{C}_k(n-1)$. Moreover, if $[a_0, a_1, \ldots, a_{n-2}, x]$ contains a negasymmetric n-tuple, where x is either 0 or $k/2$, then:*

(i) $[a_0, a_1, \ldots, a_{n-2}, x]$ *has odd period dividing c;*

(ii) $[a_0, a_1, \ldots, a_{n-2}, x]$ *contains precisely one negasymmetric n-tuple;*

(iii) $a_{c-1} = x$, $(a_0, a_1, \ldots, a_{n-1})$ *equals the concatenation of 2^t copies of* $(a_0, a_1, \ldots, a_{c-1})$, *and* $[a_0, a_1, \ldots, a_{c-1}]$ *is a negasymmetric necklace containing a single negasymmetric c-tuple.*

Proof. Since $(a_0, a_1, \ldots, a_{n-2})$ is negasymmetric we have

$$a_i = -a_{n-2-i}$$

for every i $(0 \leq i \leq n-2)$. It then follows immediately that, if $x = 0$ or $k/2$,

$$(a_0, a_1, \ldots, a_{n-2}, x) = (-a_{n-2}, -a_{n-3}, \ldots, -a_0, -x)$$

and so $[a_0, a_1, \ldots, a_{n-2}, x]$ is a negasymmetric necklace, as required.

Now suppose $[a_0, a_1, \ldots, a_{n-2}, x]$ contains a negasymmetric n-tuple where $x = 0$ or $k/2$; for notational convenience let $a_{n-1} = x$. That is, suppose

$$(a_{\overline{t}}, a_{\overline{t+1}}, \ldots, a_{\overline{t-1}}) = -(a_{\overline{t-1}}, a_{\overline{t-2}}, \ldots, a_{\overline{t}})$$

for some t, where $\overline{y} = y \bmod n$. That is

$$a_{\overline{t+i}} = -a_{\overline{t-1-i}}$$

for every i $(0 \leq i < n)$. Thus, setting $j = t + i$, we have:

$$a_j = -a_{\overline{2t-1-j}}$$

for every j $(0 \leq j < n)$.

But we know that $a_i = -a_{n-2-i}$ for every i, and hence

$$a_{\overline{-2-j}} = a_{\overline{2t-1-j}}$$

for all j $(0 \leq j < n)$ and, setting $i = -2 - j$ we get

$$a_{\overline{i}} = a_{\overline{2t+1+i}}$$

for every i ($0 \leq i < n$). Hence $[a_0, a_1, .., a_{n-1}]$ has period dividing $2t+1$, i.e. it has odd period, necessarily dividing c, and (i) follows. Result (ii) follows immediately from Lemma 2.9 given that $[a_0, a_1, .., a_{n-1}]$ has odd period.

For (iii), the fact that $a_{c-1} = x$ and $(a_0, a_1, \ldots, a_{n-1})$ equals the concatenation of 2^t copies of $(a_0, a_1, \ldots, a_{c-1})$ follows immediately from (i), i.e. from the fact that $[a_0, a_1, \ldots, a_{n-2}, a_{n-1}]$ has odd period dividing c. Finally, Lemma 2.4 means that $[a_0, a_1, \ldots, a_{c-1}]$ is a negasymmetric necklace containing a single negasymmetric c-tuple. □

Example 4.1. If $k = 3$, then (12012) is a negasymmetric 5-tuple. From the above [120120] is a negasymmetric necklace. It has period 3 and contains the negasymmetric 6-tuple (201201).

Using the results above we can now give the cardinalities of $N_i(n, k)$ for every i.

Lemma 4.8. *Suppose $k \geq 3$ and $n \geq 4$. Let $n = 2^t m$, where $t > 0$ and m is odd. Let $\delta = 1$ or 2 depending whether k is odd or even. Then:*

(i)

$$|N_0(n, k)| = \frac{\delta}{2}(\delta k^{(n-2)/2} - k^{(m-1)/2});$$

(ii)

$$|N_1(n, k)| = \delta k^{(m-1)/2},$$

and every necklace in $N_1(n, k)$ has period dividing m;

(iii)

$$|N_2(n, k)| = \frac{k^{n/2} - \delta k^{(m-1)/2}}{2}.$$

Proof. (i) It follows from Lemma 4.7 that every negasymmetric $(n-1)$-tuple $(a_0, a_1, \ldots, a_{n-2})$ is contained in δ distinct negasymmetric necklaces $[a_0, a_1, \ldots, a_{n-1}]$ in $\mathcal{C}_k(n-1)$, each of which contains either zero or one negasymmetric n-tuple(s) and has even or odd period, respectively. There are $\delta k^{(n-2)/2}$ such $(n-1)$-tuples, from Lemma 1.1

From Lemma 4.5, a negasymmetric necklace $[a_0, a_1, \ldots, a_{n-1}]$ in $\mathcal{C}_k(n-1)$ of even period containing no negasymmetric n-tuples, of which there are $|N_0(n, k)|$, contains precisely 2 distinct negasymmetric $(n-1)$-tuples.

From Lemma 4.6, a negasymmetric necklace $[a_0, a_1, \ldots, a_{n-1}]$ in $\mathcal{C}_k(n-1)$ of odd period containing one negasymmetric n-tuple, of which there are $|N_1(n, k)|$, contains a unique negasymmetric $(n-1)$-tuple.

If we count (negasymmetric $(n-1)$-tuple, necklace) pairs in two ways, where the $(n-1)$-tuple is in the necklace, we get

$$\delta^2 k^{(n-2)/2} = 2|N_0(n,k)| + |N_1(n,k)|.$$

It follows immediately (observing that $|N_1(n,k)| = \delta k^{(m-1)/2}$ from part (ii)) that

$$|N_0(n,k)| = \frac{\delta}{2}(\delta k^{(n-2)/2} - k^{(m-1)/2}).$$

(ii) From Lemma 2.9, a negasymmetric necklace contains a single negasymmetric n-tuple if and only if it has odd period. That is, a negasymmetric necklace in $N_1(n,k)$ must have period dividing m. By Lemma 1.1, the number of negasymmetric m-tuples for odd m is $\delta k^{(m-1)/2}$, where δ is as in the statement of the theorem, and so $|N_1(n,k)| = \delta k^{(m-1)/2}$.

(iii) By Lemma 1.1, there are $k^{n/2}$ negasymmetric n-tuples, since n is even. Of these, $N_1(n,k)$ appear in a negasymmetric necklace of odd period, one n-tuple per necklace. The remaining $k^{n/2} - N_1(n,k)$ negasymmetric n-tuples appear in the $|N_2(n,k)|$ negasymmetric necklaces containing two negasymmetric n-tuples each. Hence $2|N_2(n,k)| + |N_1(n,k)| = k^{n/2}$, and the result follows.

□

Using Lemmas 4.3 and 4.8, the values of $|N_i(n,k)|$ are tabulated for small values of n and k in Table 1, where the values are given as a triple for $i = 0, 1, 2$.

Table 1: Values of $|N_i(n,k)|$ for small n and k

n	$k=3$	$k=4$	$k=5$	$k=6$
3	(0,3,0)	(0,8,0)	(0,5,0)	(0,12,0)
4	(1,1,4)	(7,2,7)	(2,1,12)	(11,2,17)
5	(0,9,0)	(0,32,0)	(0,25,0)	(0,72,0)
6	(3,3,12)	(28,8,28)	(10,5,60)	(66,12,102)
7	(0,27,0)	(0,128,0)	(0,125,0)	(0,432,0)
8	(13,1,40)	(127,2,127)	(62,1,312)	(431,2,647)

4.4 The main result

We can now combine Lemmas 4.3 and 4.8 to give an upper bound on the size of $X_k(n-1)$.

Theorem 4.9. *Suppose $k \geq 3$ and $n \geq 3$. Let $n = 2^t m$ where $t > 0$ and m is odd. Let $\delta = 1$ if k is odd and $\delta = 2$ if k is even. Suppose $X_k(n-1)$ is as defined in Theorem 3.1. Then:*

$$|X_k(n-1)| \geq \max\left(\frac{k^n - r_{k,n,kn/2}}{2},\ s_k(n-1)\right)$$

where

$$s_k(n-1) = \frac{k^n - n(|N_0(n,k)| + |N_2(n,k)|) - m(|N_1(n,k)| - \delta) - \delta}{2}$$

or, equivalently,

$$s_k(n-1) = \begin{cases} \dfrac{k^n - \delta(n(k^{(n-1)/2} - 1) - 1)}{2} & \text{if } n \text{ is odd} \\ \dfrac{k^n - \frac{nk^{(n-2)/2}}{2}(\delta^2 + k) + n\delta k^{(m-1)/2} - m\delta(k^{(m-1)/2} - 1) - \delta}{2} & \text{if } n \text{ is even} \end{cases}.$$

Proof. By definition, $|X_k(n-1)| = |E_k(n-1)| + w/2$, where w is the total number of edges in the necklaces in $X_k(n-1)$ taken from $\mathcal{C}_k(n-1)$. By Lemma 4.1, w is at least

$$r_{k,n,kn/2} - n(|N_0(n,k)| + |N_2(n,k)|) - m(|N_1(n,k)| - \delta) - \delta.$$

As noted in Section 1, $|E_k(n-1)| = (k^n - r_{k,n,kn/2})/2$, and the first inequality follows.

The second inequality for the n odd case follows immediately from Lemma 4.3, observing that in this case $m = n$ and $|N_0(n,k)| = |N_2(n,k)| = 0$. If n is even, from Lemma 4.8,

$$\begin{aligned} |N_0(n,k)| + |N_2(n,k)| &= \frac{\delta}{2}(\delta k^{(n-2)/2} - k^{(m-1)/2}) + \frac{k^{n/2} - \delta k^{(m-1)/2}}{2} \\ &= \frac{k^{(n-2)/2}}{2}(\delta^2 + k) - \delta k^{(m-1)/2}, \end{aligned}$$

and

$$|N_1(n,k)| = \delta k^{(m-1)/2},$$

and the result now follows. □

The bounds of Theorem 4.9 are tabulated for small k and n in Table 2. More specifically, the values of $s_k(n-1)$ are given in the table, with the corresponding value of $\frac{k^n - r_{k,n,kn/2}}{2}$ given in brackets. That is, the actual lower bound is the larger of the two values.

Observe that the bound of Theorem 4.9 is only a lower bound on the size of $|X_k(n-1)|$, and the bound is only tight when n is prime. This is exemplified by the analyses in Examples 2.5 and 2.7 that yield sets of n-tuples slightly larger than the value given in Table 3.

Table 2: Lower bounds on the size of $X_k(n-1)$

n	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$
3	10	22	56	92	162	234	352	472
	(10)	(22)	(56)	(89)	(162)	(225)	(352)	(454)
4	30	99	284	591	1146	1955	3192	4863
	(31)	(93)	(278)	(550)	(1109)	(1835)	(3084)	(4604)
5	101	436	1502	3712	8283	16068	29324	49504
	(96)	(386)	(1432)	(3362)	(8008)	(14858)	(28624)	(46449)
6	316	1870	7596	22808	58248	129946	264520	497932
	(294)	(1586)	(7162)	(20441)	(55518)	(119895)	(254004)	(467468)
7	1002	7750	38628	138462	410574	1044998	2388936	4993006
	(897)	(6476)	(36220)	(123895)	(393991)	(965569)	(2321848)	(4697914)
8	3068	31751	193816	835495	2876916	8376327	21508784	49972007
	(2727)	(26333)	(181550)	(749422)	(2748581)	(7766075)	(20750748)	(47167644)
9	9481	128776	973754	5027192	20166003	67072008	193680724	499910008
	(8272)	(106762)	(912944)	(4526720)	(19373760)	(62405190)	(188369056)	(473247274)

5 Constructing negative orientable and orientable sequences

5.1 Negative orientable sequences

Theorem 3.1 states that $X_k(n-1)$ is an Eulerian subgraph of $B_k(n-1)$ and is antinegasymmetric. Following the same argument as employed in [12, Section 3.2], this immediately means that a directed circuit in $X_k(n-1)$, which must exist because it is Eulerian, will yield a $\mathcal{NOS}_k(n)$ of period $|X_k(n-1)|$. Hence it is possible to construct a $\mathcal{NOS}_k(n)$ with period at least that given by Theorem 4.9 — see also Table 2. Previously, the sequences with the largest known periods were derived from directed circuits in $E_k(n-1) \subseteq X_k(n-1)$, i.e. they had periods as given by the bracketed values in Table 2.

In fact, as we claimed in the abstract, these sequences are asymptotically best possible with regards to period. Let $\ell(n,k)$ denote the maximum period for a negative orientable sequence of order n over an alphabet of size k.

We first need the following trivial result.

Lemma 5.1 (Corollary 3.2 of [12]). *Suppose $n \geq 2$ and $k \geq 2$. Then*

$$\ell(n,k) \leq \begin{cases} \dfrac{k^n - \delta k^{(n-1)/2}}{2} & \text{if } n \text{ is odd} \\ \dfrac{k^n - k^{n/2}}{2} & \text{if } n \text{ is even} \end{cases}$$

where, as previously, $\delta = 1$ if k is odd and $\delta = 2$ if k is even.

Remark 5.1. Of course, sharper bounds are known (see, for example, [12, Theorem 3.5]), but this simple bound suffices for our purposes here.

Theorem 5.2. *Suppose $n \geq 3$ and $k \geq 3$. The negative orientable sequences generated using $X_k(n-1)$ have asymptotically optimal period with respect* *to both n and k. That is*

$$\frac{s_k(n-1)}{\ell(n,k)} \to 1 \quad \text{as } n \to \infty$$

and

$$\frac{s_k(n-1)}{\ell(n,k)} \to 1 \quad \text{as } k \to \infty.$$

Proof. Denote the bound of Lemma 5.1 by $b(n,k)$. First suppose n is odd. Then:

$$\begin{aligned}
\ell(n,k) - s_k(n-1) &\le b(n,k) - s_k(n-1) \\
&= \frac{k^n - \delta k^{(n-1)/2}}{2} - \frac{k^n - \delta(n(k^{(n-1)/2}-1)-1)}{2} \\
&= \delta\frac{n(k^{(n-1)/2}-1)-1) - k^{(n-1)/2}}{2} \\
&= \delta\frac{(n-1)k^{(n-1)/2} - n - 1}{2} < \delta\frac{nk^{(n-1)/2}}{2}.
\end{aligned}$$

Now, trivially, $\ell(n,k) \ge s_k(n-1)$, and hence:

$$\begin{aligned}
\frac{\ell(n,k) - s_k(n-1)}{s_k(n-1)} &< \frac{\delta\frac{nk^{(n-1)/2}}{2}}{\frac{k^n - \delta(n(k^{(n-1)/2}-1)-1)}{2}} \\
&< \frac{\delta nk^{(n-1)/2}}{k^n - \delta nk^{(n-1)/2})} \\
&\to 0 \text{ as either } n \to \infty \text{ or } k \to \infty,
\end{aligned}$$

and the result follows.

Now suppose n is even. First observe that, since $n \ge m$,

$$s_k(n-1) > \frac{k^n - \frac{nk^{(n-2)/2}}{2}(\delta^2 + k)}{2}.$$

Hence

$$b(n,k) - s_k(n-1) < \frac{\frac{nk^{(n-2)/2}}{2}(\delta^2+k) - k^{n/2}}{2},$$

and so

$$\begin{aligned}
\frac{\ell(n,k) - s_k(n-1)}{s_k(n-1)} &< \frac{\frac{nk^{(n-2)/2}}{2}(\delta^2+k) - k^{n/2}}{k^n - \frac{nk^{(n-2)/2}}{2}(\delta^2+k)} \\
&< \frac{\frac{nk^{(n-2)/2}}{2}(\delta^2+k)}{k^n - \frac{nk^{(n-2)/2}}{2}(\delta^2+k)} \\
&\to 0 \text{ as either } n \to \infty \text{ or } k \to \infty,
\end{aligned}$$

as required. $\square$

5.2 Orientable sequences

We next show how to construct a large period orientable sequence from $X_k(n-1)$; following the approach of [13], we first define the Lempel homomorphism D, mapping from $B_k(n)$ to $B_k(n-1)$. We follow Alhakim and Akinwande [1], who generalised the original Lempel definition [10] for $k = 2$, to alphabets of arbitrary size.

Definition 5.1 (Alhakim and Akinwande, [1])**.** Define the function $D_\beta : B_k(n) \to B_k(n-1)$ as follows, where $\beta \in \mathbb{Z}_k^*$. If $\mathbf{a} = (a_0, a_1, \dots, a_{n-1}) \in \mathbb{Z}_k^n$ then

$$D_\beta(\mathbf{a}) = (\beta(a_1 - a_0), \beta(a_2 - a_1), \dots \beta(a_{n-1} - a_{n-2})).$$

We are particularly interested in the case $\beta = 1$, and in what follows we simply write D for D_1. The following result is key.

Lemma 5.3 (Lemma 5.2 of [13])**.** *Suppose $n \geq 2$ and $k \geq 3$. If T is an antinegasymmetric subgraph of the de Bruijn digraph $B_k(n-1)$ with edge set E, then $D^{-1}(E)$, of cardinality $k|E|$, is the set of edges for an antisymmetric subgraph of $B_k(n)$, which, abusing our notation slightly, we refer to as $D^{-1}(T)$. Moreover, if every vertex of T has in-degree equal to its out-degree, then the same apples to $D^{-1}(T)$.*

Combining Lemma 2.5 with Theorem 3.1 immediately gives the following.

Corollary 5.4. *If $n \geq 3$ and $k \geq 3$ then $D^{-1}(X_k(n-1))$ is the set of edges of an antisymmetric subgraph of $B_k(n)$ in which every vertex of T has in-degree equal to its out-degree.*

We also have the following.

Lemma 5.5. *If $n \geq 3$ and $k \geq 3$ then $D^{-1}(X_k(n-1))$ is the set of edges for a connected subgraph of $B_k(n)$.*

Proof. Theorem 5.5 of [13] shows that the set of edges in $D^{-1}(E_k(n-1))$ forms a connected subgraph of $B_k(n)$. Thus we need only show that any edge in $D^{-1}(X_k(n-1) - E_k(n-1))$ belongs to the same connected component of $D^{-1}(X_k(n-1))$ as an edge in $D^{-1}(E_k(n-1))$. If $\mathbf{a} = (a_0, a_1, \dots, a_n)$ is such an edge, then $D(\mathbf{a})$ is in $X_k(n-1) - E_k(n-1))$. But by Theorem 3.1, $X_k(n-1)$ is Eulerian, and hence there is a linked sequence of directed edges connecting $D(\mathbf{a})$ to an edge in $E_k(n-1)$, where each directed edge links to the next edge via a common vertex. This immediately can be used, by applying D^{-1}, to construct a linked sequence of directed edges linking $\mathbf{a}$ to an edge in $D^{-1}(E_k(n-1))$ and the result follows. $\square$

It follows immediately from Corollary 5.4 and Lemma 5.5 that any directed circuit in $D^{-1}(X_k(n-1))$ will yield an $\mathcal{OS}_k(n+1)$. We immediately have the following.

Theorem 5.6. *If $n \geq 3$ and $k \geq 3$ then there exists an $\mathcal{OS}_k(n+1)$ of period $ks_k(n-1)$, where $s_k(n-1)$ is as specified in Theorem 4.9.*

The lower bounds on the periods of the sequences of Theorem 5.6 for small k and n are given in Table 3, along with the periods of the sequences from [13, Table 5] in brackets for comparison.

Table 3: Periods of the constructed $\mathcal{OS}_k(n)$ (and bounds)

n	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
4	30	88	280	552	1134	1872
	(30)	(88)	(280)	(534)	(1134)	(1800)
5	90	396	1420	3546	8022	15640
	(93)	(372)	(1390)	(3300)	(7763)	(14680)
6	303	1744	7510	22272	57981	128544
	(288)	(1544)	(7160)	(20172)	(56056)	(118864)
7	948	7480	37980	136848	407736	1039568
	(882)	(6344)	(35810)	(122646)	(388626)	(959160)
8	3006	31000	193140	830772	2874018	8359984
	(2691)	(25904)	(181100)	(743370)	(2757937)	(7724552)

It is important to note that Theorem 5.6 only gives a lower bound on the period of the orientable sequences obtained by the approach described in this paper, and the bound is only tight when $n-1$ is prime. As previously noted in connection with Table 2, this is exemplified by the analyses in Examples 2.5 and 2.7 that yield sequences with a slightly larger period than that given in Table 3.

Finally, as for the negative orientable case, we show that the sequences of Theorem 5.6 have asymptotically optimal period. Let $\lambda(n,k)$ denote the maximum period for an orientable sequence of order n over an alphabet of size k. The following result is elementary.

Lemma 5.7 (Lemma 16 of [4])**.** *Suppose $n \geq 2$ and $k \geq 2$. Then*

$$\lambda(n,k) \leq \begin{cases} \dfrac{k^n - k^{(n+1)/2}}{2} & \text{if } n \text{ is odd} \\ \dfrac{k^n - k^{n/2}}{2} & \text{if } n \text{ is even} \end{cases}.$$

Remark 5.2. As with Lemma 5.1, sharper bounds are known (see, for example, [13, Theorem 2.13]), but this simple bound suffices here.

Theorem 5.8. *Suppose $n \geq 3$ and $k \geq 3$. The orientable sequences of Theorem 5.6 have asymptotically optimal period with respect to both n and k. That is*

$$\frac{ks_k(n-1)}{\lambda(n+1,k)} \to 1 \quad \textit{as } n \to \infty$$

and

$$\frac{ks_k(n-1)}{\lambda(n+1,k)} \to 1 \quad \textit{as } k \to \infty.$$

Proof. Denote the bound of Lemma 5.7 by $c(n,k)$. First suppose n is odd. Then:

$$\begin{aligned}
\lambda(n+1,k) - ks_k(n-1) &\leq c(n+1,k) - ks_k(n-1) \\
&= \frac{k^{n+1} - k^{(n+1)/2}}{2} - \frac{k^{n+1} - \delta k(n(k^{(n-1)/2}-1)-1)}{2} \\
&= \frac{\delta k(n(k^{(n-1)/2}-1)-1) - k^{(n+1)/2}}{2} \\
&= \frac{\delta n k^{(n+1)/2} - \delta k(n+1) - k^{(n+1)/2}}{2} < \frac{\delta n k^{(n+1)/2}}{2}.
\end{aligned}$$

Now, trivially, $\lambda(n+1,k) \geq ks_k(n-1)$, and hence:

$$\begin{aligned}
\frac{\lambda(n+1,k) - ks_k(n-1)}{ks_k(n-1)} &< \frac{\delta n k^{(n+1)/2}}{k^{n+1} - \delta k(n(k^{(n-1)/2}-1)-1)} \\
&< \frac{\delta n k^{(n+1)/2}}{k^{n+1} - \delta n k^{(n+1)/2})} \\
&\to 0 \text{ as either } n \to \infty \text{ or } k \to \infty,
\end{aligned}$$

and the result follows.

Now suppose n is even. First observe that, since $n \geq m$,

$$ks_k(n-1) > \frac{k^{n+1} - \frac{nk^{n/2}}{2}(\delta^2 + k)}{2}.$$

Hence

$$c(n+1,k) - ks_k(n-1) < \frac{\frac{nk^{n/2}}{2}(\delta^2+k) - k^{(n+2)/2}}{2},$$

and so

$$\begin{aligned}
\frac{\lambda(n+1,k)-ks_k(n-1)}{ks_k(n-1)} &< \frac{\frac{nk^{n/2}}{2}(\delta^2+k)-k^{(n+2)/2}}{k^{n+1}-\frac{nk^{n/2}}{2}(\delta^2+k)} \\
&< \frac{\frac{nk^{n/2}}{2}(\delta^2+k)}{k^{n+1}-\frac{nk^{n/2}}{2}(\delta^2+k)} \\
&\to 0 \text{ as either } n\to\infty \text{ or } k\to\infty,
\end{aligned}$$

as required. □

6 Concluding remarks

It is important to assess how the construction method described here affects efforts to find orientable sequences with the largest possible period. The current state of knowledge for small n and $k>2$ is summarised in Table 4, where the upper bound from [13, Theorem 2.13] is given in brackets beneath the largest known period.

Table 4: Largest known periods for an $\mathcal{OS}_k(n)$ (and bounds)

n	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
2	**3**	**4**	**10**	**12**	**21**	**24**
	(3)	(4)	(10)	(12)	(21)	(24)
3	**9**	**20**	**50**	**84**	**147**	**216**
	(9)	(20)	(50)	(84)	(147)	(216)
4	**30**	88	**280**	552	**1134**	1872
	(30)	(112)	(280)	(612)	(1134)	(1984)
5	93	404	1420	3546	8022	15640
	(99)	(452)	(1450)	(3684)	(8085)	(15896)
6	303	1744	7510	22272	57981	128544
	(315)	(1958)	(7550)	(23019)	(58065)	(130332)
7	954	7480	37980	136848	407736	1039568
	(972)	(7844)	(38100)	(138144)	(408072)	(1042712)
8	3006	31000	193140	830772	2874018	8359984
	(3096)	(32390)	(193800)	(837879)	(2876496)	(8382492)

The following observations can be made about this table. The values in bold represent maximal values.

- $n=2$: The fact that there exists an $\mathcal{OS}_k(2)$ with period meeting the bound follows from [2, Theorem 5.4] (for k prime), and from [9, Theorem 2] and [12, Lemma 2.2] for general k.

- $n = 3$: The existence of an $\mathcal{OS}_k(3)$ meeting the period bound is due to [2, Example 5.1] for $k = 3$, [9, Section 1] for $k = 5$ (from an exhaustive search), and for general k from [13].

- $n = 4$: The fact that the maximum period of an $\mathcal{OS}_3(4)$ is 30 is again due to [9, Section 1], and the existence of an $\mathcal{OS}_k(4)$ meeting the bound for general odd k is from [13]. the values for $k = 6$ and $k = 8$ follow from this paper.

- $n = 5$: The $\mathcal{OS}_3(5)$ of period 93 comes from [13]. The values for $k > 3$ come from this paper. The fact that the value for $n = 5$ and $k = 4$ is 404, and not 396 as guaranteed by the bound of Theorem 5.6, arises from the analysis in Example 2.7.

- $n \geq 6$: All values in the table come from this paper. Analogously to the case $n = 5$ and $k = 4$, the value of 954, and not 948, for $n = 7$ and $k = 3$ arises from the analysis in Example 2.5.

That is, the results in this paper appear to yield sequences with larger period than were previously known for all values of $n \geq 5$. In particular, the values in the table exceed those obtained by Gabrić and Sawada [9], whose construction technique also yields sequences with asymptotically optimal period. As has been observed above, except when $n - 1$ is prime, the value yielded by the bound of Theorem 5.6 is typically less than the value actually obtained.

However, there are many questions that remain to be resolved. We briefly mention two here.

- First, and as noted in Remark 2.1, we have focussed in this paper on adding edges from non-negasymmetric necklaces in $\mathcal{C}_k(n-1)$ to $E_k(n-1)$. However, if k is even then $\mathcal{C}_k(n-1)$ is by no means the only way of dividing the n-tuples of pseudoweight $kn/2$ into circuits, and there may be other ways of doing this that enable a larger set of n-tuples to be added to $E_k(n-1)$ while preserving antinegasymmetry and the Eulerian property.

- Second, whilst the approach described in this paper results in orientable and negative orientable sequences with period greater than any previously known sequences, it is far from clear whether or not sequences with greater period can be constructed.

That is, while progress has been achieved, the search for optimal period orientable sequences is not over yet!

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