arXiv:2603.25286v3 [math.CO] 9 Nov 2026

Negative Avoiding Sequences

Chris J Mitchell and Peter R Wild

9th November 2026

Abstract

Negative avoiding sequences of span n are periodic sequences of elements from $\mathbb{Z}_k$ for some k with the property that no n-tuple occurs more than once in a period and if an n-tuple does occur then its negative does not. They are a special type of cut-down de Bruijn sequence with potential position-location applications. We establish a simple upper bound on the period of such a sequence, and refer to sequences meeting this bound as *maximal* negative avoiding sequences. We then go on to demonstrate the existence of maximal negative avoiding sequences for every $k \geq 3$ and every $n \geq 2$.

1 Introduction

The main focus of this paper is a special class of infinite periodic sequences with elements from $\mathbb{Z}_k$, for some $k > 2$ — which we refer to as k-ary sequences. We are concerned with n-tuples (or *factors*), i.e. ordered strings of n symbols, that are equal to n consecutive symbols from a sequence. We refer to an n-tuple *occurring* in a sequence if the n-tuple equals n consecutive symbols occurring somewhere in the sequence; analogously we refer to a sequence *containing* an n-tuple if that n-tuple occurs somewhere in the sequence. For a sequence $S = (s_i)$ and $n \geq 1$ we write $\mathbf{s}_n(i) = (s_i, s_{i+1}, \ldots, s_{i+n-1})$, for the n-tuple corresponding to a substring or factor of n consecutive symbols occurring in the sequence at positions $i, i+1, \ldots, i+n-1$.

If a sequence $S = (s_i)$ has period m then we represent the sequence as $[s_0, s_1, \ldots, s_{m-1}]$, i.e. a single period of the sequence contained in square brackets. In much of the literature $[s_0, s_1, \ldots, s_{m-1}]$ is described as a *cycle*.

We refer to the sequences studied here as *Negative Avoiding Sequences (NASs)*. Informally, a Negative Avoiding Sequence of span n is a periodic sequence of elements from $\mathbb{Z}_k$ for some $k > 2$, with two special properties:

- every possible n-tuple of length n occurs at most once in a period of the sequence; and

- if an n-tuple $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ occurs in the sequence, then the n-tuple $(-a_0, -a_1, \ldots, -a_{n-1})$ does not occur in the sequence.

In this paper we provide a simple bound on the period of a NAS, and give constructions establishing that the bound is tight for all $n \geq 2$ and $k \geq 3$.

More formally, we can make the following definitions.

Definition 1.1 ([3])**.** A k-ary *n-window sequence* $S = (s_i)$ is a periodic sequence of elements from $\mathbb{Z}_k$ ($k > 1$, $n > 1$) with the property that no n-tuple occurs more than once in a period of the sequence, i.e. with the property that if $\mathbf{s}_n(i) = \mathbf{s}_n(j)$ for some i, j, then $i \equiv j \pmod{m}$ where m is the period of the sequence.

Remark 1.1. An n-window sequence is also known as a cut-down de Bruijn Sequence — see, for example, Jackson et al. [14, Problem 477] and Cameron et al. [6].

Note that the well-studied *de Bruijn sequences* of span n (see, for example, [16, Section 7.2.1.1]) are simply n-window sequences in which every possible n-tuple occurs exactly once in a period. Such sequences clearly have period k^n.

We now make the following key definition.

Definition 1.2. Suppose $k \geq 3$ and $n \geq 1$. If $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ is a k-ary n-tuple, then we define its negative in the obvious way, i.e.

$$-\mathbf{a} = (-a_0, -a_1, \ldots, -a_{n-1}).$$

The special class of n-window sequences of interest here can now be defined.

Definition 1.3. Suppose $k \geq 3$ and $n \geq 2$. A k-ary *Negative Avoiding Sequence (NAS)* of span n is an infinite periodic n-window sequence $S = (s_i)$, $i \geq 0$, with elements from $\mathbb{Z}_k$ with the property that if the n-tuple $\mathbf{a}$ occurs in S, then $-\mathbf{a}$ does not occur in S. We refer to such a sequence a $\mathcal{NAS}_k(n)$.

If a k-ary n-tuple $\mathbf{a}$ satisfies $\mathbf{a} = -\mathbf{a}$ then we call it *self-negative*. Clearly a $\mathcal{NAS}_k(n)$ cannot contain any self-negative n-tuples.

Example 1.1. A simple example for $k = 3$ and $n = 2$ would be the cycle [0112], in which the 2-tuples 01, 11, 12, 20 appear, but their negatives, i.e. 02, 22, 21 and 10 do not. Since 00 cannot appear being self-negative, this sequence clearly has maximum possible period.

The remainder of the paper is structured as follows. Motivation for the work and background definitions are provided in Sections 2 and 3. This is followed in Section 4 by an upper bound on the period of negative avoiding sequences. Results establishing the existence of maximal period negative avoiding sequences are provided in Sections 5, 6 and 7. The paper concludes in Section 8 with a summary and a discussion of open research questions.

2 Motivation and related work

De Bruijn sequences have a range of real-world applications, ranging from position location on a linear track or rotating shaft [4, 5, 22, 24, 30], to genome assembly [8]. As observed by Cameron et al. [6], 'for some applications it may be more convenient to produce a cycle of arbitrary length such that there are no repeated strings', i.e. an n-window sequence of a particular period that derives from the specifics of the application.

A number of authors have considered the problem of constructing n-window sequences with particular properties. Jackson et al. [14, Problem 477] observed that a binary n-window sequence exists for every possible period m satisfying $n \leq m \leq 2^n$. Sawada et al. [27] and Ruskey et al. [26] showed that a binary n-window sequence can be constructed containing all n-tuples with weights in the range $[a, b]$ as long as $0 \leq a < b \leq n$, and such sequences were further studied by Li et al. [18]. Later, Gabrić et al. [11], generalised this result to alphabets of an arbitrary size in the case where weight is defined as the digit sum, and showed that all n-tuples bounded above (or below) by a given weight can be arranged in a cut-down de Bruin sequence. Even more recently, Gabrić et al. [10] showed how to construct sequences of this type that can be readily decoded.

Sequences containing only n-tuples with specified weights are a particular example of what are sometimes known as 'de Bruijn sequences with forbidden subsequences', as for example examined in the binary case by Penne [23] and for arbitrary alphabets by Alhakim [1] and Sawada et al. [28]. Tan and Shallot [29], studied the more general problem of the existence of sequences containing only the members of a pre-specified set of n-tuples, where they allow such n-tuples to occur multiple times in a period. These are all examples of *Universal cycles* [7], i.e. periodic sequences in which all elements of a particular set of n-tuples occur precisely once in a period.

The sequences we consider in this paper are closely related to, yet distinct from, those previously examined. That is, we consider the case where the n-tuples are divided into pairs (an n-tuple and its negative) and we only permit at most one of each pair to occur in a sequence. Such a sequence might have application in a position-location case where it is difficult for a reader to distinguish between an n-tuple and its negative. These sequences are closely related to *orientable* [9, 21] and *negative orientable* sequences [19], which are also periodic n-window sequences in which the n-tuples are divided into pairs and at most one of each pair is permitted to occur in a sequence; the only difference is how the 'pairing' function is defined.

3 The de Bruijn graph and the Lempel homomorphism

Before proceeding we need to introduce certain key concepts which we use in the remainder of the paper.

3.1 The de Bruijn graph

Following Alhakim et al. [3], for positive integers n and k greater than one, let $\mathbb{Z}_k^n$ be the set of all k^n tuples of length n with entries from the group $\mathbb{Z}_k$ of residues modulo k.

Definition 3.1. The order n de Bruijn digraph, $B_k(n)$, is a directed graph with $\mathbb{Z}_k^n$ as its vertex set in which, for any two vertices $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$ and $\mathbf{y} = (y_0, y_1, \ldots, y_{n-1})$, the pair $(\mathbf{x}, \mathbf{y})$ is an edge if and only if $y_i = x_{i+1}$ for every i $(0 \leq i < n-1)$. We label such an edge with the $(n+1)$-tuple $(x_0, x_1, \ldots, x_{n-1}, y_{n-1})$.

Note that we have defined two ways of specifying an edge in $B_k(n)$, namely as either a pair of vertices $(\mathbf{x}, \mathbf{y})$, where $\mathbf{x}, \mathbf{y}$ are k-ary n-tuples, or as a single k-ary $(n+1)$-tuple $\mathbf{z}$.

Definition 3.2. An *Eulerian digraph* is a connected digraph for which every vertex has in-degree equal to out-degree — this latter property characterises a *balanced* digraph.

The name derives from the fact that there exists an Eulerian circuit, i.e. a path visiting every edge once, in a digraph if and only if the digraph is Eulerian — see, for example, Corollary 6.1 of Gibbons [12] or Knuth [15, Section 2.3.4.2, Theorem G]. Moreover, there are simple and efficient algorithms for finding Eulerian circuits — see for example [12, Figure 6.5] or [15, Section 2.3.4.2, Theorem D].

3.2 Antinegative subgraphs

It is straightforward to verify that there is a correspondence between a k-ary de Bruijn sequence of order n and a directed Eulerian circuit in $B_k(n-1)$, in which consecutive edges in the circuit correspond to consecutive n-tuples occurring in the sequence. Inspired by this correspondence, and following [21], we next describe a correspondence between negative avoiding sequences of span n and Eulerian circuits in subgraphs of $B_k(n-1)$, where these subgraphs satisfy certain special properties. If we can then construct subgraphs with these special properties which admit Eulerian circuits then we have a

simple method of generating negative avoiding sequences. Note that the graphs we are concerned with here are examples of Rauzy graphs [25].

Definition 3.3. Suppose T is a subgraph of the de Bruijn digraph $B_k(n)$ for some $n \geq 2$ and $k \geq 2$. T is said to be *antinegative* if, given vertices $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1}), \mathbf{y} = (y_0, y_1, \ldots, y_{n-1}) \in \mathbb{Z}_k^n$ where $(\mathbf{x}, \mathbf{y})$ is an edge in T, then $(-\mathbf{x}, -\mathbf{y})$ is *not* an edge in T.

Definition 3.4 (Definition 3.2 of [21])**.** Suppose S is a periodic k-ary n-window sequence of period m for some $n \geq 2$, $k \geq 2$ and $m \geq 1$. Then the *edge-graph* $E_n(S)$ of S is defined to be the subgraph of $B_k(n-1)$ whose directed edges correspond to n-tuples occurring in a period of S, i.e. with edge set

$$\{\mathbf{s}_n(i) \ : \ 0 \leq i \leq m-1\}$$

and whose vertices are those vertices of $B_k(n-1)$ that have in-degree at least one.

We can now state a key lemma.

Lemma 3.1. *Suppose S is a k-ary periodic n-window sequence. Then S is an $\mathcal{NAS}_k(n)$ if and only if $E_n(S)$ is antinegative in $B_k(n-1)$.*

Proof. Suppose S is an $\mathcal{NAS}_k(n)$. Then, by definition, $\mathbf{s}_n(i) \neq -\mathbf{s}_n(j)$, for any i, j. Hence, again by definition, this means that $\mathbf{a} \neq -\mathbf{b}$ for any edges $\mathbf{a}, \mathbf{b}$ in $E_n(S)$. Hence $E_n(S)$ is antinegative.

Now suppose $E_n(S)$ is antinegative, and hence $\mathbf{a} \neq -\mathbf{b}$ for any edges $\mathbf{a}, \mathbf{b}$ in $E_n(S)$. Thus, $\mathbf{s}_n(i) \neq -\mathbf{s}_n(j)$, for any i, j, and so S is an $\mathcal{NAS}_k(n)$. □

This enables us to give a result which is key to establishing the results in the remainder of the paper.

Theorem 3.2. *If S is an $\mathcal{NAS}_k(n)$ of period m then $E_n(S)$ is an antinegative Eulerian subgraph of $B_k(n-1)$ containing m edges. Moreover, if T is an antinegative Eulerian subgraph of $B_k(n-1)$ with m edges, then there exists an $\mathcal{NAS}_k(n)$ S of period m such that $E_n(S) = T$.*

Proof. Suppose $S = (s_i)$ is an $\mathcal{NAS}_k(n)$ of period m. Then, by Lemma 3.1, $E_n(S)$ is antinegative in $B_k(n-1)$. Also, trivially, $E_n(S)$ contains m edges. Finally, if $(x_1, x_2, \ldots, x_{n-1})$ is a vertex in $E_n(S)$, then the incoming edges all have distinct labels of the form $(x, x_1, x_2, \ldots, x_{n-1})$ for some x, and correspond to $\mathbf{s}_n(i)$ for some i. Then the edges corresponding to $\mathbf{s}_n(i+1)$ will have distinct labels of the form $(x_1, x_2, \ldots, x_{n-1}, y)$ for some y, and for every incoming edge there is an outgoing edge, and vice versa. Hence the in-degree of every vertex is the same as the out-degree, i.e. $E_n(S)$ is balanced. Finally,

note that $E_n(S)$ is connected since we only include vertices in $E_n(S)$ with in-degree greater than zero, and the sequence S defines a path incorporating every vertex in $E_n(S)$.

Now suppose T is an antinegative Eulerian subgraph of $B_k(n-1)$ with m edges. Then there exists an Eulerian circuit of length m in T corresponding to a sequence S with period m in the natural way. S is clearly an n-window sequence since the circuit visits every edge exactly once, and each edge corresponds to a unique n-tuple. It is straightforward to verify that $E_n(S) = T$. Finally, S is an $\mathcal{NAS}_k(n)$ from Lemma 3.1. □

Remark 3.1. In general, for any antinegative Eulerian subgraph of $B_k(n-1)$ T, there may exist many negative avoiding sequences with edge-graph T, since there may be many different Eulerian circuits in T, each corresponding to a different sequence.

3.3 Implications

Theorem 3.2 means that if we can construct an antinegative Eulerian subgraph of $B_k(n-1)$ with m edges for 'large' m (i.e. with m close to the maximum possible), then we will immediately have a set of negative avoiding sequences with period close to the maximum. As a result, in the remainder of this paper we consider ways of constructing sets of edges in $B_k(n-1)$ that define an antinegative Eulerian subgraph of $B_k(n-1)$.

3.4 The Lempel Homomorphism

We next define the Lempel homomorphism D. Following [21] we give a simplified version of the definition of Alhakim and Akinwande [2], who generalised the original Lempel definition [17].

Definition 3.5. Let $D : \mathbb{Z}_k^n \to \mathbb{Z}_k^{n-1}$ be defined such that

$$D(\mathbf{a}) = (a_1 - a_0, a_2 - a_1, \ldots, a_{n-1} - a_{n-2})$$

where $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1}) \in \mathbb{Z}_k^n$. Clearly D is onto; thus we also define D^{-1} to map an element of $\mathbb{Z}_k^{n-1}$ to the set of its pre-images under D.

We have the following.

Lemma 3.3 (Alhakim & Akinwande, [2])**.** $|D^{-1}(\mathbf{a})| = k$ *for every* $\mathbf{a} \in \mathbb{Z}_k^{n-1}$.

Since the edges of $B_k(n-1)$ are associated with k-ary n-tuples, D induces a mapping, which we also write as D, from $B_k(n)$ to $B_k(n-1)$. It is simple to show that D is a graph homomorphism. In this context we also write D^{-1}

for the pre-image mapping which maps an edge of $B_k(n-1)$ to a set of k edges of $B_k(n)$ and each pre-image edge contains corresponding pre-image vertices.

The following simple result shows the value of D in constructing negative avoiding sequences.

Lemma 3.4. *Suppose $n \geq 2$ and $k \geq 3$. If T is an antinegative subgraph of the de Bruijn digraph $B_k(n-1)$ with edge set E, then $D^{-1}(E)$, of cardinality $k|E|$, is the set of edges for an antinegative subgraph $D^{-1}(T)$ of $B_k(n)$. Moreover, the in-degree (out-degree) of any vertex $\mathbf{x}$ in $D^{-1}(T)$ equals the in-degree (out-degree) of $D(\mathbf{x})$ in T; thus, in particular, if T is balanced then so is $D^{-1}(T)$.*

Proof. Suppose $D^{-1}(T)$ is not antinegative, i.e. there exist $(n+1)$-tuples $\mathbf{a} = (a_0, a_1, \ldots, a_n), \mathbf{b} = (b_0, b_1, \ldots, b_n) \in D^{-1}(E)$ such that $\mathbf{a} = -\mathbf{b}$, i.e. $a_i = -b_i$, for $0 \leq i \leq n$.

Suppose also that $D(\mathbf{a}) = \mathbf{c}$ and $D(\mathbf{b}) = \mathbf{d}$, where $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1}), \mathbf{d} = (d_0, d_1, \ldots, d_{n-1}) \in E$. Hence $c_i = a_{i+1} - a_i$ and $d_i = b_{i+1} - b_i$ for $0 \leq i \leq n-1$. Since $a_i = -b_i$ for $0 \leq i \leq n$, for any j $(0 \leq j \leq n-1)$ we have:

$$c_j = a_{j+1} - a_j = -b_{j+1} + b_j = -d_j.$$

Hence $\mathbf{c} = -\mathbf{d}$, but this contradicts the assumption that T is antinegative.

Every edge in E corresponds to k edges in $D^{-1}(E)$, and hence $|D^{-1}(E)| = k|E|$.

It remains to show that every vertex of $D^{-1}(T)$ has in-degree equal to its out-degree. Suppose $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is a vertex of $D^{-1}(T)$. For every edge $\mathbf{a} = (a, x_1, x_2, \ldots, x_n) \in D^{-1}(T)$ that ends in $\mathbf{x}$, there is a corresponding edge $D(\mathbf{a}) = (x_1 - a, x_2 - x_1, \ldots, x_n - x_{n-1}) \in T$ that ends in $D(\mathbf{x})$. That is, the in-degree of $\mathbf{x}$ in $D^{-1}(T)$ will equal the in-degree of $D(\mathbf{x})$ in T. An exactly similar result holds for out-degree. Since every vertex of T has in-degree equal to its out-degree, the same holds for $D^{-1}(T)$. □

4 A simple bound and some examples

Having established our key underlying ideas, we give the following elementary bound.

Theorem 4.1. *Suppose $k \geq 3$ and $n \geq 2$. If S is a $\mathcal{NAS}_k(n)$ of period m then*

$$m \leq \begin{cases} \frac{k^n-1}{2} & \text{if } k \text{ is odd} \\ \frac{k^n-2^n}{2} & \text{if } k \text{ is even} \end{cases}$$

Proof. Clearly, if an n-tuple $\mathbf{a}$ is self-negative, i.e. it satisfies $\mathbf{a} = -\mathbf{a}$, then it cannot occur in a $\mathcal{NAS}_k(n)$. There is one such n-tuple if k is odd, namely $(0, 0, \ldots, 0)$, and 2^n if k is even, namely all n-tuples with entries equal to 0 or $k/2$. □

Definition 4.1. An $\mathcal{NAS}_k(n)$ ($n \geq 2$, $k \geq 3$) is said to be *maximal* if it has period meeting the bound of Theorem 4.1.

In the remainder of this paper we are concerned with determining the existence of maximal negative avoiding sequences for all possible $n \geq 2$ and $k \geq 3$. We give three examples showing that such sequences exist for small values of n and k.

Example 4.1. If $k = 3$ and $n = 2$, then $[0, 1, 1, 2]$ is a maximal $\mathcal{NAS}_3(2)$, since it has period $(3^2 - 1)/2 = 4$.

If $k = 4$ and $n = 2$, then $[1, 3, 0, 1, 2, 1]$ is a maximal $\mathcal{NAS}_4(2)$, since it has period $(4^2 - 2^2)/2 = 6$.

If $k = 3$ and $n = 3$ then $[0, 1, 0, 0, 1, 1, 1, 0, 1, 2, 1, 1, 2]$ is a maximal $\mathcal{NAS}_3(3)$, since it has period $(3^3 - 1)/2 = 13$.

Before proceeding observe that, if $k \geq 3$ and $n \geq 2$, it follows immediately from the definition that if $[a_0, a_1, \ldots, a_{m-1}]$ is a $\mathcal{NAS}_k(n)$, then so is $[-a_0, -a_1, \ldots, -a_{m-1}]$. Hence if $[a_0, a_1, \ldots, a_{m-1}]$ is a maximal $\mathcal{NAS}_k(n)$, then $[a_0, a_1, \ldots, a_{m-1}]$ and $[-a_0, -a_1, \ldots, -a_{m-1}]$ contain all the non-self-negative n-tuples. The self-negative tuples can be arranged to form a sequence of period 1 or 2^n, depending on whether k is odd or even. These three sequences can be merged to generate a de Bruijn sequence since the de Bruijn graph is strongly connected.

5 Maximal sequences for odd n

We next establish the existence of a maximal $\mathcal{NAS}_k(n)$ for any odd $n \geq 3$ and any $k \geq 3$.

5.1 Preliminaries

We first need the following concept.

Definition 5.1 (pseudoweight, [19])**.** Suppose $\mathbf{u} = (u_0, u_1, \ldots, u_{n-1})$ is an n-tuple of elements of $\mathbb{Z}_k$ ($n \geq 1$, $k \geq 3$). Define the function $f : \mathbb{Z}_k \to \mathbb{Q}$ as follows: for any $u \in \mathbb{Z}_k$ treat u as an integer in the range $[0, k-1]$ and set $f(u) = u$ if $u \neq 0$ and $f(u) = k/2$ if $u = 0$. Then the *pseudoweight* of $\mathbf{u}$ is

defined to be the sum

$$w^*(\mathbf{u}) = \sum_{i=0}^{n-1} f(u_i)$$

where the sum is computed in $\mathbb{Q}$.

As a simple example for $k = 3$, the 4-tuple $(0, 1, 1, 2)$ has weight $0+1+1+2 = 4$ and pseudoweight $1.5 + 1 + 1 + 2 = 5.5$, since $f(0) = \frac{3}{2}$.

Definition 5.2 ([21])**.** Suppose $n \geq 2$ and $k \geq 3$. Let $E_k(n)$ be the set of all k-ary n-tuples with pseudoweight less than $nk/2$.

We can now give a key result.

Theorem 5.1 ([21])**.** *Suppose $n \geq 2$ and $k \geq 3$. $E_k(n)$ is the set of edges for an Eulerian subgraph $U_k(n-1)$, of the de Bruijn digraph $B_k(n-1)$.*

We also have the following simple result, which follows immediately given that, for any k-ary n-tuple $\mathbf{a}$, $w^*(\mathbf{a}) = nk - w^*(-\mathbf{a})$.

Lemma 5.2. *Suppose $n \geq 2$ and $k \geq 3$. $U_k(n-1)$ is an antinegative subgraph of the de Bruijn digraph $B_k(n-1)$.*

Example 5.1 (cf. Example 5.2 of [21])**.** As an example, consider the case $k = 3$ and $n = 3$. The ten 3-ary 3-tuples having pseudoweight less than 4.5 are

$$111,\ 011,\ 101,\ 110,\ 001,\ 010,\ 100,\ 112,\ 121,\ 211,$$

forming the set $E_3(3)$. These can readily be arranged to form a $\mathcal{NAS}_3(3)$ of period 10 (less than the maximum possible value which is 13), e.g.:

$$[1110010112].$$

While the above analysis immediately shows how to construct an antinegative Eulerian subgraph for any k and n (and hence a $\mathcal{NAS}_k(n)$ from Theorem 3.2), the sequences will not be maximal, since they do not contain any n-tuples with weight exactly $kn/2$. We next show how, if n is odd, we can rectify this. We first need to introduce a little more notation.

Definition 5.3 ([20])**.** Suppose $n \geq 2$ and $k \geq 3$. Let $H_k(n-1)$ be the subgraph of $B_k(n-1)$ restricted to edges of pseudoweight precisely $kn/2$.

5.2 The construction

Following [20], we present a simple way of dividing the edges (n-tuples) in $H_k(n-1)$ into edge-disjoint circuits. Let $(a_0, a_1, \ldots, a_{n-1})$ be a k-ary n-tuple, i.e. an edge in $B_k(n-1)$. Let m be the smallest positive integer c

such that $a_i = a_{\overline{i+c}}$ for every i $(0 \leq i < n)$, where $\overline{x} = x \bmod n$. We write $[a_0, a_1, \ldots, a_{n-1}]$ for the circuit in $B_k(n-1)$ consisting of edges

$$(a_0, a_1, \ldots, a_{n-1}), (a_1, a_2 \ldots, a_{n-1}, a_0), \ldots, (a_{m-1}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{m-2}),$$

and say the circuit has period m.

Lemma 5.3 ([20])**.** *Suppose $k \geq 3$ and $n \geq 3$.*

i) If $(a_0, a_1, \ldots, a_{n-1})$ is a k-ary n-tuple of pseudoweight $kn/2$, i.e. an edge in $H_k(n-1)$, then $[a_0, a_1, \ldots, a_{n-1}]$ is a circuit in $H_k(n-1)$ of period dividing n.

ii) Let $\mathcal{C}_k(n-1)$ be the set of all circuits of the form $[a_0, a_1, \ldots, a_{n-1}]$, where $(a_0, a_1, \ldots, a_{n-1})$ is an edge in $H_k(n-1)$; then $\mathcal{C}_k(n-1)$ forms an edge-disjoint partition of the edges in $H_k(n-1)$.

Observe that if a circuit contains a self-negative n-tuple, then all the n-tuples in the circuit are self-negative, and we refer to the circuit as being self-negative.

Definition 5.4. Let $\mathcal{C}_k^*(n-1)$ be the set of non-self-negative circuits in $\mathcal{C}_k(n-1)$.

The following observation is key.

Lemma 5.4. *Suppose $k \geq 3$ and $n \geq 3$ is odd. No circuit in $\mathcal{C}_k^*(n-1)$ can contain both an n-tuple and its negative.*

Proof. Suppose not, i.e. suppose

$$(a_0, a_1, \ldots, a_{n-1}) = -(a_{\overline{j}}, a_{\overline{j+1}}, \ldots, a_{\overline{j-1}})$$

where $\overline{j} = j \bmod n$. Hence

$$a_i = (-1)^s a_{\overline{sj+i}}$$

for every i $(0 \leq i < n)$ and every $s \geq 0$. Setting $s = n$ and observing that n is odd immediately gives

$$a_i = -a_i$$

for every i $(0 \leq i < n)$, and hence $(a_0, a_1, \ldots, a_{n-1})$ is self-negative, giving a contradiction. □

This immediately gives the following.

Lemma 5.5. *Suppose $k \geq 3$ and $n \geq 3$ is odd. The circuits in $\mathcal{C}_k^*(n-1)$ can be divided into pairs, consisting of a circuit and its negative.*

We can now describe our simple construction.

Construction 5.1. Suppose $k \geq 3$ and $n \geq 3$ is odd. Select one of each pair of circuits in $\mathcal{C}_k^*(n-1)$, as per Lemma 5.5, and let $F_k(n)$ be the set of k-ary n-tuples containing all the n-tuples in $E_k(n)$ together with all the n-tuples occurring in the selected circuits from $\mathcal{C}_k^*(n-1)$. Let $W_k(n-1)$ be the subgraph of $B_k(n-1)$ whose edges correspond to n-tuples in $F_k(n)$ and whose vertices belong to these edges.

Lemma 5.6. *Suppose $k \geq 3$ and $n \geq 3$ is odd. Then $W_k(n-1)$ is an antinegative Eulerian subgraph of $B_k(n-1)$, and $|F_k(n)| = \frac{k^n - \delta^n}{2}$, where $\delta = 1$ or 2 depending on whether k is odd or even.*

Proof. From Theorem 5.1 and Lemma 5.2 we know that $E_k(n)$ is the set of edges for antinegative Eulerian subgraph of $B_k(n-1)$. All the edges in $E_k(n)$ have pseudoweight less than $kn/2$, and hence since they only contain edges of weight exactly $kn/2$, neither an n-tuple nor its negative from a circuit in $\mathcal{C}_k^*(n-1)$ can equal an n-tuple in $E_k(n)$; moreover, since we only selected one circuit of each pair from $\mathcal{C}_k^*(n-1)$, it follows that $F_k(n)$ is the set of edges for an antinegative subgraph of $B_k(n-1)$.

If r is the number of k-ary n-tuples of pseudoweight exactly $kn/2$, then clearly $|E_k(n)| = \frac{k^n - r}{2}$. Moreover, since the circuits in $\mathcal{C}_k^*(n-1)$ cover all the non-self-negative n-tuples of pseudoweight exactly $kn/2$, the number of n-tuples occurring in a circuit in $\mathcal{C}_k^*(n-1)$ is $r - \delta^n$. Since $F_k(n)$ contains precisely half the n-tuples occurring in a circuit in $\mathcal{C}_k^*(n-1)$,

$$|F_k(n)| = |E_k(n)| + \frac{r - \delta^n}{2} = \frac{k^n - r}{2} + \frac{r - \delta^n}{2} = \frac{k^n - \delta^n}{2}.$$

It remains to show that the subgraph is Eulerian, i.e. it is balanced and connected. Since $U_k(n-1)$ is Eulerian (from Theorem 5.1), and since we are adding complete circuits to $E_k(n)$, it follows that the subgraph is balanced. Moreover, every circuit in $\mathcal{C}_k^*(n-1)$ must contain an edge whose corresponding n-tuple contains at least one element in the range $[k/2, k-1]$, and hence it must contain an edge on a vertex whose corresponding $(n-1)$-tuple has pseudoweight at most $(n-1)k/2$. Now every vertex whose corresponding $(n-1)$-tuple has pseudoweight at most $(n-1)k/2$ has ingoing and outgoing edges in $E_k(n)$, and hence the subgraph with edge set $F_k(n)$ is connected. □

Combining Lemma 5.6 with Theorems 3.2 and 4.1 immediately gives the following.

Theorem 5.7. *If $k \geq 3$ and $n \geq 3$ is odd then there exists a maximal $\mathcal{NAS}_k(n)$.*

Example 5.2 (cf. Example 2.1 of [20])**.** Suppose $n = k = 3$. The seven edges in $H_3(2)$ can be partitioned into three circuits:

$$[000], [012], [021]$$

of periods 1, 3 and 3, respectively. Thus $\mathcal{C}_3^*(2) = \{[012], [021]\}$, i.e. it contains one pair of a circuit and its negative. Choosing one, e.g., [012], and adding it to the $\mathcal{NAS}_3(3)$ of Example 5.1, i.e. [1110010112], we obtain the following maximal $\mathcal{NAS}_3(3)$ of period 13:

$$[1110012010112].$$

5.3 The n even case is not so simple

Things are less straightforward if n is even, since in this case a circuit in $\mathcal{C}_k^*(n-1)$ *can* contain both an n-tuple and its negative. Indeed, the following simple example means that, for n even, we cannot create maximal NASs simply by taking all tuples of pseudoweight less than $nk/2$ and then adding in selected tuples of weight exactly $nk/2$.

Example 5.3. Suppose $k = 3$ and $n = 4$. If a maximal $\mathcal{NAS}_3(4)$ exists, then it cannot only contain 4-tuples with pseudoweight less than or equal to $kn/2 = 6$. To see why this is true, suppose that S is a maximal $\mathcal{NAS}_3(4)$ (i.e. one with period 40) containing only 4-tuples with pseudoweight less than or equal to 6. It must contain all 31 4-tuples of pseudoweight less than 6, and half the 18 4-tuples with pseudoweight exactly 6.

Next observe that S cannot contain a 3-tuple of the form $(2, 1, 2)$. If S did contain an occurrence of $(2, 1, 2)$ then such a 3-tuple must be preceded and succeeded by 1 within S, given that S only contains 4-tuples of pseudoweight at most 6. However, if S contains the 5-tuple $(1, 2, 1, 2, 1)$ then S will contain the 4-tuples $(1, 2, 1, 2)$ and $(2, 1, 2, 1)$, which gives a contradiction since they are the negative of each other. Hence S cannot contain any 3-tuples of the form $(2, 1, 2)$. Now consider the 4-tuples $(1, 2, 1, 2)$ and $(2, 1, 2, 1)$ of weight 6. S must contain one of them — which immediately gives a contradiction since they both contain the 3-tuple $(2, 1, 2)$.

6 Maximal sequences for $n = 2$

As a first step towards demonstrating the existence of maximal negative avoiding sequences for all even n, we start by looking at $n = 2$. We consider the k odd and even cases separately.

6.1 Maximal sequences for $n = 2$ and k odd

Definition 6.1. Suppose $k \geq 3$ is odd. Let $Y_k \subseteq \mathbb{Z}_k^2$ be the following set of k-ary 2-tuples:

$$\{(x, y) : \overline{y - x} \in \{1, 2, \ldots, (k-1)/2\}\} \cup \{(x, x) : x \in \{1, 2, \ldots, (k-1)/2\}\},$$

where $\overline{m}$ denotes m mod k.

Lemma 6.1. *Suppose $k \geq 3$ is odd. Then:*

(i) $|Y_k| = \frac{k^2-1}{2}$;

(ii) the 2-tuples in Y_k correspond to the set of edges of an Eulerian subgraph of $B_k(1)$;

(iii) if $(x, y) \in Y_k$ then $(\overline{-x}, \overline{-y}) \notin Y_k$.

Proof. For each of the k possible values for x, there are precisely $(k-1)/2$ values y for which $\overline{y - x} \in \{1, 2, \ldots, (k-1)/2\}$. Hence $|Y_k| = \frac{k(k-1)}{2} + \frac{k-1}{2} = \frac{k^2-1}{2}$, and (i) follows.

Consider any vertex (x) in $B_k(1)$. If $x \in \{1, 2, \ldots, (k-1)/2\}$ then (x) has in-degree and out-degree $(k+1)/2$ in the subgraph defined by Y_k; otherwise (x) has in-degree and out-degree $(k-1)/2$. That is every vertex has its in-degree equal to its out-degree. To show it is Eulerian it remains to show that it is connected. Suppose (x) and (y) are distinct vertices. If $\overline{y - x} \in \{1, 2, \ldots, (k-1)/2\}$ then $(x, y) \in Y_k$; otherwise $(y, x) \in Y_k$, and hence (ii) follows.

Finally, if $x \neq y$ then $(x, y) \in Y_k$ implies $\overline{y - x} \in \{1, 2, \ldots, (k-1)/2\}$, and hence $\overline{x - y} \notin \{1, 2, \ldots, (k-1)/2\}$, i.e. $(\overline{-x}, \overline{-y}) \notin Y_k$. If $x = y$ then $(x, x) \in Y_k$ implies $x \in \{1, 2, \ldots, (k-1)/2\}$, and hence $\overline{-x} \notin \{1, 2, \ldots, (k-1)/2\}$, i.e. $(\overline{-x}, \overline{-x}) \notin Y_k$, and (iii) follows. □

Since, from Lemma 6.1(ii), the 2-tuples in Y_k correspond to the set of edges of an Eulerian subgraph of $B_k(1)$, it is possible to define a sequence S in which each of the 2-tuples in Y_k occurs exactly once. Such a sequence is a $\mathcal{NAS}_k(2)$ from Lemma 6.1(iii). Moreover, from Lemma 6.1(i), S has period $|Y_k| = \frac{k^2-1}{2}$. This immediately yields the following from Theorems 3.2 and 4.1.

Theorem 6.2. *If $k \geq 3$ is odd there exists a maximal $\mathcal{NAS}_k(2)$.*

Example 6.1. Suppose $k = 3$. Then

$$Y_3 = \{(0, 1),\ (1, 2),\ (2, 0),\ (1, 1)\}.$$

These can be joined to create a maximal $\mathcal{NAS}_3(2)$ of period 4, namely: $[0, 1, 1, 2]$ (cf. Example 4.1).

6.2 Maximal sequences for $n = 2$ and k even

Construction 6.1. Suppose $k \geq 4$ is even. Observe that $U_k(1)$, the Eulerian subgraph of $B_k(1)$ with edge set $E_k(2)$, contains (amongst others) the edges $(0, i), (i, i)$ for every i, $0 < i < k/2$. Let $U'_k(1)$ be the subgraph of $B_k(1)$ obtained by removing these edges from $U_k(1)$ and adjoining the edges $(0, -i), (-i, -i), (-i, i)$ for every i, $0 < i < k/2$.

Lemma 6.3. *If $k \geq 4$ is even then $U'_k(1)$ is an antinegative Eulerian subgraph of $B_k(1)$ containing $(k^2 - 4)/2$ edges.*

Proof. Making the specified modifications to $U_k(1)$ maintains the number of in- and out-edges to the vertex with label 0, reduces the number of in- and out- edges to vertex i, $0 < i < k/2$, by 1, but not to zero, and increases the number of in- and out- edges to vertex $-i$, $0 < i < k/2$, by 2. The subgraph $U'_k(1)$ is therefore Eulerian as now vertex $k - 1$ is connected to vertices 0 and 1, and vertex 1 remains connected to all other vertices. It is antinegative as the adjoined edges are not negatives of the existing edges or of each other. Since two edges have been removed and three edges adjoined for each i, $0 < i < k/2$, $U'_k(1)$ has $(k^2 - k - 2)/2 + (k - 2)/2 = (k^2 - 4)/2$ edges. □

We then immediately have the following result from Theorems 3.2 and 4.1.

Theorem 6.4. *If $k \geq 4$ is even there exists a maximal $\mathcal{NAS}_k(2)$.*

Example 6.2. Suppose $k = 4$. Then $U_4(1)$ contains five edges with labels:

$$\{(0, 1), (1, 0), (1, 1), (1, 2), (2, 1)\}.$$

Creating $U'_4(1)$ involves removing edges $(0, 1)$ and $(1, 1)$ and inserting edges $(0, 3)$, $(3, 3)$ and $(3, 1)$, i.e., $U'_4(1)$ contains the following edges:

$$\{(0, 3), (1, 0), (3, 3), (3, 1), (1, 2), (2, 1)\}.$$

These yield a maximal $\mathcal{NAS}_4(2)$ of period 6, e.g., $[0, 3, 3, 1, 2, 1]$.

7 Maximal sequences for even $n \geq 4$

To construct maximal negative avoiding sequences of even order we apply the pre-image mapping D^{-1} to the subgraph $W_k(n-1)$ of Construction 5.1 for n odd and adjoin to the resulting set of pre-image edges certain additional edges, as described below. We begin with a lemma about $D^{-1}(W_k(n-1))$.

Lemma 7.1. *Suppose $n \geq 3$ and $k \geq 3$, where n is odd. The subgraph $D^{-1}(W_k(n-1))$ of $B_k(n)$ has the following properties:*

(i) The number of edges of $D^{-1}(W_k(n-1))$ *is* $\frac{k^{n+1}-k\delta^n}{2}$*;*

(ii) $D^{-1}(W_k(n-1))$ *is antinegative;*

(iii) $D^{-1}(W_k(n-1))$ *is an Eulerian subgraph of the de Bruijn graph* $B_k(n)$*.*

Proof. Property (i) follows from Lemma 3.3. Property (ii) follows because D^{-1} preserves antinegativity. $W_k(n-1)$ is balanced so $D^{-1}(W_k(n-1))$ has this property too. By Theorem 5.5 of [21] $D^{-1}(U_k(n-1))$ is a connected subgraph of the de Bruijn graph $B_k(n)$. Since each circuit of $\mathcal{C}_k^*(n-1)$ shares a vertex with $U_k(n-1)$ it follows that $D^{-1}(W_k(n-1))$ is also connected. Hence (iii) holds. □

Let $T_k(n-1)$ be the subgraph of $B_k(n-1)$ whose edges correspond to all the self-negative n-tuples, i.e., edges corresponding to n-tuples with all entries equal to 0 or $k/2$, and whose vertices are those on such edges. When k is odd, $T_k(n-1)$ consists of one edge on one vertex (a loop). When k is even $T_k(n-1)$ consists of 2^n edges on 2^{n-1} vertices and is isomorphic to $B_2(n-1)$. So $T_k(n-1)$ is Eulerian. Let $\mathbf{a}$ be an Eulerian circuit in $T_k(n-1)$. Then $\mathbf{a}$ corresponds to a sequence $[a_0, a_1, \ldots, a_{m-1}]$ where $m = 1$ if n is odd and $m = 2^n$ if n is even, and $a_0 = a_1 = \cdots = a_{n-1} = 0$. Now, since $b_{m-1} = \sum_{i=0}^{m-1} a_i = 0$, as $k/2$ occurs an even number of times, $D^{-1}(\mathbf{a})$ can be partitioned into k circuits $\mathbf{b}_x$, $x \in \mathbb{Z}_k$, of $B_k(n)$ corresponding to sequences $[x, x+a_0, x+a_0+a_1, \ldots, x+\sum_{i=0}^{m-2} a_i] = [x, x+b_0, x+b_1, \ldots, x+b_{m-2}]$, where $b_i = \sum_{j=0}^{i} a_j$, $i = 0, \ldots, m-2$.

Let $Z_k(n)$ be the subgraph of $B_k(n)$ whose edges are the edges of $D^{-1}(W_k(n-1))$ and the edges of $\mathbf{b}_x$ for $0 < x < k/2$.

Theorem 7.2. *Suppose* $k \geq 3$ *and* $n \geq 3$ *is odd. Then* $Z_k(n)$ *is an Eulerian subgraph of* $B_k(n)$ *and there exists a maximal* $\mathcal{NAS}_k(n+1)$*.*

Proof. Each edge of $D^{-1}(W_k(n-1))$ has as its image under D an edge which is not self-negative, and each edge of $\mathbf{b}_x$ has an image which is self-negative. So $D^{-1}(W_k(n-1))$ and $\mathbf{b}_x$ are edge-disjoint for every x. Moreover, if $x_1 \neq x_2$ then $\mathbf{b}_{x_1}$ and $\mathbf{b}_{x_2}$ do not share an edge, since if $(x_1+b_i, x_1+b_{i+1}, \ldots, x_1+b_{i+n}) = (x_2+b_j, x_2+b_{j+1}, \ldots, x_2+b_{j+n})$ then either $i = j$ and $x_1 = x_2$ (a contradiction) or $i \neq j$ and $(a_i, a_{i+1}, \ldots, a_{i+n-1}) = (a_j, a_{j+1}, \ldots, a_{j+n-1})$ (a contradiction). By Lemma 7.1(i) the number of edges of $Z_k(n)$ is

$$\frac{k^{n+1}-k\delta^n}{2} + \frac{k-\delta}{2}\delta^n = \frac{k^{n+1}-\delta^{n+1}}{2},$$

where, as before, $\delta = 1$ if k is odd and $\delta = 2$ if k is even, since each circuit $\mathbf{b}_x$ contains δ^n edges.

By Lemma 7.1(iii) and the fact that each circuit $\mathbf{b}_x$ is Eulerian, $\mathbb{Z}_k(n-1)$ is Eulerian since $D^{-1}(W_k(n-1))$ shares a vertex with each $\mathbf{b}_x$, e.g., $(x, x, \ldots, x)$ which is a vertex of both $\mathbf{b}_x$ and $D^{-1}(W_k(n-1))$, as $(x, 0, \ldots, 0,)$, having pseudoweight less than $nk/2$, belongs to $W_k(n-1)$.

Now $D^{-1}(W_k(n-1))$ is antinegative from Lemma 7.1(ii). Since $-a_i = a_i$, $i = 0, 1, \ldots, m-1$, it follows that $-(x + b_i) = -x + b_i$, $i = 0, 1, \ldots, m-1$, and as $k/2 < -x < k$ when $0 < x < k/2$ the negative of an edge of any $\mathbf{b}_x$, $0 < x < k/2$, does not belong to $Z_k(n)$. Thus $Z_k(n)$ is antinegative. Hence there exists an Eulerian circuit in $Z_k(n)$, which corresponds to an $\mathcal{NAS}_k(n+1)$ of period $\frac{k^{n+1}-\delta^n}{2}$, i.e., a maximal $\mathcal{NAS}_k(n+1)$. □

The above theorem completes the project of demonstrating the existence of a maximal $\mathcal{NAS}_k(n)$ for every $n \geq 2$ and every $k \geq 3$.

Example 7.1. As an example of the above construction, suppose $k = n = 3$. From Examples 5.1 and 5.2, we can set $W_3(2)$ to be the subgraph containing the 13 edges corresponding to the following 3-tuples:

$$111, 011, 101, 110, 001, 010, 100, 112, 121, 211, 012, 120, 201.$$

Then $D^{-1}(W_3(2))$ contains the following 39 4-tuples:

$$0120, 0012, 0112, 0122, 0001, 0011, 0111, 0121, 0101, 0201, 0010, 0100, 0220,$$
$$1201, 1120, 1220, 1200, 1112, 1122, 1222, 1202, 1212, 1012, 1121, 1211, 1001,$$
$$2012, 2201, 2001, 2011, 2220, 2200, 2000, 2010, 2020, 2120, 2202, 2022, 2112.$$

$Z_3(3)$ contains one additional edge, namely $\mathbf{a}_1 = 1111$, giving a set of 40 4-tuples that can be joined to create a maximal $\mathcal{NAS}_3(4)$, e.g.:

$$[0001012120220201001201220112112001111222]$$

.

Example 7.2. As a second example of the above construction, suppose $k = 4$ and $n = 3$.

The set $E_4(3)$ of 3-tuples over $\mathbb{Z}_4$ of pseudoweight less than $nk/2 = 6$ has $|E_4(3)| = 22$ elements, as follows:

$$E_4(3) = \{\, 001, 010, 011, 012, 021, 100, 101, 102, 110, 111,$$
$$112, 113, 120, 121, 122, 131, 201, 210, 211, 212, 221, 311 \,\}.$$

The set $H_4(2)$ of 3-tuples of pseudoweight exactly $nk/2 = 6$ has $|H_4(2)| = 20$ elements, which Lemma 5.3 partitions into eight circuits:

$$[000], \quad [002] = \{002, 020, 200\}, \quad [022] = \{022, 202, 220\}, \quad [222],$$

all four of which are self-negative, and

$$[013] = \{013, 130, 301\}, \quad [031] = \{031, 103, 310\},$$
$$[123] = \{123, 231, 312\}, \quad [132] = \{132, 213, 321\},$$

which are not. Thus

$$C_4^*(2) = \{[013], [031], [123], [132]\},$$

and under negation we obtain the pairs $\{[013], [031]\}$ and $\{[123], [132]\}$. Selecting one circuit from each pair, say [013] and [123], adjoin the six 3-tuples $013, 130, 301, 123, 231, 312$.

Set $W_4(2)$ to be the set of edges in

$$F_4(3) = E_4(3) \cup \{013, 130, 301\} \cup \{123, 231, 312\},$$

so $|F_4(3)| = 22 + 3 + 3 = 28 = (4^3 - 2^3)/2$, as required by Lemma 5.6. Then $W_4(2)$ is antinegative and balanced, and is hence an antinegative Eulerian subgraph of $B_4(2)$ (and incidentally yields a maximal $NAS_4(3)$ of period 28).

For each edge $e = (e_0, e_1, e_2) \in W_4(2)$ and each $x \in \mathbb{Z}_4$ we obtain a 4-tuple

$$\big(x,\ x + e_0,\ x + e_0 + e_1,\ x + e_0 + e_1 + e_2\big).$$

This produces $D^{-1}(W_4(2))$, with

$$|D^{-1}(W_4(2))| = 4 \times 28 = 112 = \frac{4^4 - 4 \cdot 2^3}{2},$$

as per Lemma 7.1(i); it is antinegative and balanced by Lemma 7.1(ii),(iii).

$T_4(2)$ consists of the eight self-negative 3-tuples over $\mathbb{Z}_4$ (with entries in $\{0, 2\}$) and is isomorphic to $B_2(2)$. Taking the binary de Bruijn sequence $[0, 0, 0, 1, 0, 1, 1, 1]$ in $B_2(2)$ and mapping $0 \mapsto 0$, $1 \mapsto 2$, gives the Eulerian circuit

$$\mathbf{a} = E_n([0, 0, 0, 2, 0, 2, 2, 2]).$$

where the elements sum to 0 in $\mathbb{Z}_4$.

Since $k/2 = 2$, we adjoin $\mathbf{b}_x$ only for $x = 1$. Setting $\mathbf{b}_1 = E_{n+1}([1, 1, 1, 1, 3, 3, 1, 3])$ gives eight additional edges

$$A = \{1111,\ 1113,\ 1133,\ 1331,\ 3313\ 3131,\ 1311,\ 3111\}.$$

Combining, we get

$$Z_4(3) = D^{-1}(W_4(2)) \ \cup \ A\},$$

where

$$|Z_4(3)| = 112 + 8 = 120 = \frac{4^4 - 2^4}{2},$$

and $Z_4(3)$ is antinegative, balanced, and contains no self-negative 4-tuple. An example of an Eulerian circuit through the 120 edges of $Z_4(3)$ is the following maximal $NAS_4(4)$ of period 120:

$$[0,0,0,1,3,3,1,3,1,2,3,3,3,0,3,3,0,2,3,3,1,2,2,3,3,0,1,3,2,3,$$
$$1,3,0,2,2,3,2,3,0,3,0,2,1,2,3,2,2,3,1,2,0,3,0,1,2,3,1,1,2,3,$$
$$0,2,0,1,3,1,1,1,3,3,0,0,2,3,1,0,1,3,0,1,1,3,0,0,1,2,2,2,3,0,$$
$$1,0,1,2,1,2,0,1,2,0,0,1,0,0,1,1,2,2,0,1,1,1,1,2,1,1,2,0,2,3].$$

8 Conclusions and open questions

We have introduced a special category of cut-down de Bruijn sequences, the negative avoiding sequences, and gave an upper bound on their period. We then showed by construction that the bound is tight for all $n \geq 2$ and $k \geq 3$.

The sequences under study are an example of a special category of cut-down de Bruijn sequences, as are also the orientable and negative orientable sequences. For $n \geq 2$ and $k \geq 2$, these sequences can be characterised as follows.

Definition 8.1. Suppose p is a self-inverse permutation (i.e., satisfying $p^{-1} = p$) on the set of k-ary n-tuples, i.e., on the set $(\mathbb{Z}_k)^n$. Then a *pair-selecting sequence* for the permutation p is a k-ary cut-down de Bruijn sequence of span n with the property that if the n-tuple $\mathbf{a}$ occurs in the sequence then $p(\mathbf{a})$ does not.

It is simple to see that orientable, negative orientable and negative avoiding sequences are all examples of pair-selecting sequences for appropriate choices of p:

- for a negative avoiding sequence, p is defined so that $p(\mathbf{a}) = -\mathbf{a}$;
- for an orientable sequence, p is defined so that $p(\mathbf{a}) = \mathbf{a}^R$;
- for a negative orientable sequence, p is defined so that $p(\mathbf{a}) = -\mathbf{a}^R$;

where for any n-tuple $\mathbf{a}$, $\mathbf{a}^R$ denotes the n-tuple obtained by reversing the order of its entries. It should thus be clear that these three categories of sequence are closely related in some sense, although, unlike the case for negative avoiding sequences, the problem of determining the maximum possible period of orientable and negative orientable sequences remains unresolved for all but small n (see, for example, [20]).

More generally, one might ask whether it is possible to obtain more general results on such sequences applying for large classes of permutation p. One

obvious result is that the upper bound on the period of such a sequence will always be $\frac{k^n-f^n}{2}$, where f is the number of fixed points of the permutation p. Of course, this bound will not always be sharp, as is demonstrated by orientable and negative orientable sequences for which smaller upper bounds can be derived.

A special case of this question would be where the permutation p derives from a self-inverse permutation of the elements of $\mathbb{Z}_k$, as is the case for negative avoiding sequences. It would interesting to know whether the trivial upper bound on the period is always achievable in this case.

The simplest case of this is when $k = 2$ and p simply switches 0 to 1 and vice versa; in this case there are no fixed points, i.e., $f = 0$, and hence a maximal sequence has period 2^{n-1}. If S is such a maximal sequence, then $p(S)$, i.e., S with every bit switched, is clearly also a maximal sequence, and between them S and $p(S)$ contain every possible n-tuple exactly once in a period. We refer to such a pair of maximal sequences as a *complementary pair*.

If n is odd then a simple construction involves the set of binary n-tuples of Hamming weight less than $n/2$. That this set defines an Eulerian subgraph of $B_2(n-1)$ was established by Sawada at al. [27], as observed in Section 2. The size of the set is clearly 2^{n-1} and hence it can be used to create a maximal period sequence.

However, for n even, life is not quite so simple. No such sequence exists for $n = 2$ and, for $n = 4$, [11110010] is a unique example of a maximal sequence (unique up to reversal and switching zeros and ones). Sequences for every $n \geq 4$ can be derived using the inverse Lempel homomorphism, as discussed — using different terminology — by Lempel [17, Section III]. In fact, as per [17, Corollary 1], there is a 1-1 correspondence between even weight de Bruijn sequences of span n and complementary pairs of sequences with span $n + 1$, and, of course, every de Bruijn sequence of span greater than 1 has even weight. This point is discussed by Hartke [13].

Acknowledgements

We would like to thank the anonymous referee who provided a number of important comments that significantly improved this paper.

References

[1] A. Alhakim, *Designing preference functions for de Bruijn sequences with forbidden words*, Des. Codes Cryptogr. **90** (2022), no. 10, 2319–

2335.

[2] A. Alhakim and M. Akinwande, *A recursive construction of nonbinary de Bruijn sequences*, Des. Codes Cryptogr. **60** (2011), no. 2, 155–169.

[3] A. Alhakim, C. J. Mitchell, J. Szmidt, and P. R. Wild, *Orientable sequences over non-binary alphabets*, Cryptogr. Commun. **16** (2024), 1309–1326.

[4] J. Burns and C. J. Mitchell, *Coding schemes for two-dimensional position sensing*, Tech. Report HPL–92–19, January 1992, `https://www.chrismitchell.net/HPL-92-19.pdf`.

[5] ———, *Coding schemes for two-dimensional position sensing*, Cryptography and Coding III (M. J. Ganley, ed.), Oxford University Press, 1993, pp. 31–66.

[6] B. Cameron, A. Gündoğan, and J. Sawada, *Cut-down de Bruijn sequences*, Discrete Math. **348** (2025), 114024.

[7] F. Chung, P. Diaconis, and R. Graham, *Universal cycles for combinatorial structures*, Discrete Math. **110** (1992), 43–59.

[8] P. E. C. Compeau, P. A. Pevzner, and G. Tesler, *How to apply de Bruijn graphs to genome assembly*, Nat. Biotechnol. **29** (2011), 987–991.

[9] Z.-D. Dai, K. M. Martin, M. J. B. Robshaw, and P. R. Wild, *Orientable sequences*, Cryptography and Coding III (M. J. Ganley, ed.), Oxford University Press, Oxford, 1993, pp. 97–115.

[10] D. Gabric, W. Imam, L. Janik-Jones, and J. Sawada, *Decoding universal cycles for t-subsets and t-multisets by decoding bounded-weight de Bruijn sequences*, CoRR **abs/2603.11934** (2026).

[11] D. Gabric, J. Sawada, A. Williams, and D. Wong, *A successor rule framework for constructing k-ary de Bruijn sequences and universal cycles*, IEEE Trans. Inf. Theory **66** (2020), no. 1, 679–687.

[12] A. Gibbons, *Algorithmic graph theory*, Cambridge University Press, Cambridge, 1985.

[13] S. G. Hartke, *Binary de Bruijn cycles under different equivalence relations*, Discret. Math. **215** (2000), 93–102.

[14] B. Jackson, B. Stevens, and G. Hurlbert, *Research problems on Gray codes and universal cycles*, Discrete Math. **309** (2009), no. 17, 5341–5348.

[15] D. E. Knuth, *The art of computer programming, Volume 1: Fundamental algorithms*, 3rd ed., Addison-Wesley, Reading, Mass., 1997.

[16] ———, *The art of computer programming, Volume 4A: Combinatorial algorithms, Part 1*, Addison-Wesley, Reading, Mass., 2011.

[17] A. Lempel, *On a homomorphism of the de Bruijn graph and its application to the design of feedback shift registers*, IEEE Trans. Comput. **C-19** (1970), 1204–1209.

[18] M. Li, Y. Jiang, and D. Lin, *Properties of the cycles that contain all vectors of weight $\leq k$*, Des. Codes Cryptogr. **91** (2023), no. 1, 221–239.

[19] C. J. Mitchell and P. R. Wild, *Orientable and negative orientable sequences*, Discrete Appl. Math. **377** (2025), 242–259.

[20] ———, *Constructing orientable and negative orientable sequences with asymptotically optimal period*, 2026, Available at `https://arxiv.org/abs/2603.18646`.

[21] ———, *New orientable sequences*, Acta Inform. **63** (2026), 14.

[22] K.-Y. Peng, H.-S. Hsiao, and J.-Y. Chang, *Length-adaptive linear position sensing system based on de-Bruijn sequence*, 2023 IEEE SENSORS, Vienna, Austria, October 29 – Nov. 1, 2023, IEEE, 2023, pp. 1–4.

[23] R. Penne, *A note on certain de Bruijn sequences with forbidden subsequences*, Discrete Math. **310** (2010), no. 4, 966–969.

[24] E. M. Petriu, *Absolute position measurement using pseudo-random binary encoding*, IEEE Instrum. Meas. Mag. **1** (1998), no. 3, 19–23.

[25] G. Rauzy, *Suites à termes dans un alphabet fini*, Seminaire de Théorie des Nombres de Bordeaux **12** (1982–1983), 1–16.

[26] F. Ruskey, J. Sawada, and A. Williams, *De Bruijn sequences for fixed-weight binary strings*, SIAM J. Discrete Math. **26** (2012), no. 2, 605–617.

[27] J. Sawada, B. Stevens, and A. Williams, *De Bruijn sequences for the binary strings with maximum density*, WALCOM: Algorithms and Computation — 5th International Workshop, WALCOM 2011, New Delhi, India, February 18–20, 2011. Proceedings (N. Katoh and A. Kumar, eds.), Lecture Notes in Computer Science, vol. 6552, Springer, 2011, pp. 182–190.

[28] J. Sawada, A. Williams, and D. Wong, *Generalizing the classic greedy and necklace constructions of de Bruijn sequences and universal cycles*, Electron. J. Comb. **23** (2016), no. 1, 1.

[29] S. Tan and J. O. Shallit, *Sets represented as the length-n factors of a word*, Combinatorics on Words — 9th International Conference, WORDS 2013, Turku, Finland, September 16–20. Proceedings (J. Karhumäki, A. Lepistö, and L. Q. Zamboni, eds.), Lecture Notes in Computer Science, vol. 8079, Springer, 2013, pp. 250–261.

[30] K. Yamaguchi and H. Imai, *Periodic sequences for absolute type shaft encoders*, Applied Algebra, Algebraic Algorithms and Error-Correcting Codes, 8th International Conference, AAECC-8, Tokyo, Japan, August 20–24, 1990. Proceedings (S. Sakata, ed.), Lecture Notes in Computer Science, vol. 508, Springer, Berlin, Heidelberg, 1991.