# New orientable sequences

Chris J. Mitchell and Peter R. Wild
Information Security Group, Royal Holloway, University of London
me@chrismitchell.net;     peterrwild@gmail.com

3rd March 2026

**Abstract**

Orientable sequences of order $n$ are infinite periodic sequences with symbols drawn from a finite alphabet of size $k$ with the property that any tuple of $n$ elements or its reverse occurs at most once as a contiguous subsequence (i.e. a substring or factor) in a period. They were introduced in the early 1990s in the context of possible applications in position sensing. Bounds on the period of such sequences and a range of methods of construction have been devised, although apart from very small cases a significant gap remains between the largest known period for such a sequence and the best known upper bound. In this paper we first give improved upper bounds on the period of such sequences. We then give a new general method of construction for orientable sequences involving subgraphs of the de Bruijn graph with special properties, and describe two different approaches for generating such subgraphs. This enables us to construct orientable sequences with periods meeting the improved upper bounds when $n$ is 2 or 3, as well as $n = 4$ and $k$ odd. For $4 \leq n \leq 8$, in some cases the sequences produced by the methods described have periods larger than for any previously known sequences.

## 1 Introduction

### 1.1 Orientable sequences

Orientable sequences were introduced in the early 1990s [3, 4, 5] in the context of possible applications in position sensing. An orientable sequence of order $n$ is an infinite periodic sequence with symbols drawn from a finite alphabet — typically $\mathbb{Z}_k$ for some $k$ — with the property that in any period a given $n$-tuple $(a_0, \ldots, a_{n-1}) \in \mathbb{Z}_k$ or its reverse $(a_{n-1}, \ldots, a_0)$ arises as a contiguous subsequence (i.e. a substring or factor) of length $n$ at most once. That is, if anyone reading the sequence *in either direction* observes $n$ consecutive symbols, they can deduce both the direction in which they are reading and their position within one period of the sequence. Gabrić and

Sawada [8] provide an interesting discussion of further possible applications as well as their relationship to strings relevant to DNA computing.

The early work referred to above focussed on the binary case, i.e. where $k = 2$; Dai et al. [5] provided both an upper bound on the period for this case and a method of construction yielding sequences with asymptotically optimal periods. More recently, Mitchell and Wild [11] showed how the Lempel homomorphism [10] could be applied to recursively generate binary orientable sequences with periods a large fraction of the optimal value. In 2024, Gabrić and Sawada [6, 7] described a highly efficient method of generating binary orientable sequences with the largest known periods.

In 2024, Alhakim et al. [2] studied the general alphabet case, i.e. where $k > 2$. They gave an upper bound on the period of orientable sequences for all $n$ and $k$, and also described a range of methods of construction using the Alhakim and Akinwande generalisation of the Lempel homomorphism to arbitrary finite alphabets [1]. Since then a range of construction methods have been proposed [8, 12, 13]; of particular interest is the method of Gabrić and Sawada [8], who showed how to construct sequences with asymptotically optimal period for any $n$ and $k > 2$ using a cycle-joining approach.

In this paper we first give new upper bounds on the period of an orientable sequence, which improve on the previous bounds for $n \geq 3$. We also describe a novel general approach to the construction of orientable sequences for any $k$ and $n$. This approach involves showing, using graph-theoretic arguments, that a subset of $n$-tuples with special properties can always be 'joined' to create an orientable sequence. Two methods for generating sets of $n$-tuples with the appropriate set of properties are described. The orientable sequences obtained have optimal periods for $n = 2$, $n = 3$ and $n = 4$ when $k$ is odd. For $4 \leq n \leq 8$, for some values of $k$ the sequences generated have longer period than any previously known sequences, although sequences generated by the method of Gabrić and Sawada [8] have the largest known period for larger values of $n$.

## 1.2 Basic definitions

In this paper we consider periodic sequences $(s_i)$ with elements from $\mathbb{Z}_k$ for some $k$, which we refer to as $k$-ary. For a sequence $S = (s_i)$ and $n \geq 1$ we write $\mathbf{s}_n(i) = (s_i, s_{i+1}, \ldots, s_{i+n-1})$, for an $n$-tuple corresponding to a substring or factor of $n$ consecutive symbols occurring in the sequence at positions $i, i+1, \ldots, i+n-1$. We say such an $n$-tuple appears in the sequence. Since we are interested in strings occurring either forwards or backwards in a sequence, we also introduce the notion of a reversed tuple, so that if $\mathbf{u} = (u_0, u_1, \ldots, u_{n-1})$ is a $k$-ary $n$-tuple then $\mathbf{u}^R = (u_{n-1}, u_{n-2}, \ldots, u_0)$ is its *reverse*. We are also interested in negating all the elements of a tuple, and hence if $\mathbf{u} = (u_0, u_1, \ldots, u_{n-1})$ is a $k$-ary $n$-tuple, we write $-\mathbf{u}$ for $(-u_0, -u_1, \ldots, -u_{n-1})$.

**Definition 1.1** ([2]). A $k$-ary *n-window sequence* $S = (s_i)$ is a periodic sequence of elements from $\mathbb{Z}_k$ ($k > 1$, $n > 1$) with the property that no $n$-tuple appears more than once in a period of the sequence, i.e. with the property that if $\mathbf{s}_n(i) = \mathbf{s}_n(j)$ for some $i, j$, then $i \equiv j \pmod{m}$ where $m$ is the period of the sequence.

A $k$-ary de Bruijn sequence *of order $n$* is then simply an $n$-window sequence in which every $k$-ary $n$-tuple appears once in a period, i.e. an $n$-window sequence of maximal period.

Following Alhakim et al. [2] we also introduce the de Bruijn digraph. For positive integers $n$ and $k$ greater than one, let $\mathbb{Z}_k^n$ be the set of all $k^n$ tuples of length $n$ with entries from the group $\mathbb{Z}_k$ of residues modulo $k$. The order $n$ de Bruijn digraph, $B_k(n)$, is a directed graph with $\mathbb{Z}_k^n$ as its vertex set in which, for any two vertices $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$ and $\mathbf{y} = (y_0, y_1, \ldots, y_{n-1})$, the pair $(\mathbf{x}, \mathbf{y})$ is an edge if and only if $y_i = x_{i+1}$ for every $i$ ($0 \le i < n - 1$). We label such an edge with the $(n+1)$-tuple $(x_0, x_1, \ldots, x_{n-1}, y_{n-1})$.

Note that we have defined two ways of specifying an edge in $B_k(n)$, namely as either a pair of vertices $(\mathbf{a}, \mathbf{b})$, where $\mathbf{a}, \mathbf{b}$ are $k$-ary $n$-tuples, or as a single $k$-ary $(n+1)$-tuple $\mathbf{x}$. Note that, in this case, the $(n+1)$-tuple $\mathbf{x}^R$ denotes the same edge as $(\mathbf{b}^R, \mathbf{a}^R)$.

**Definition 1.2.** An *Eulerian digraph* is a connected digraph for which every vertex has in-degree equal to out-degree.

The name derives from the fact that there exists an Eulerian circuit, i.e. a circuit visiting every edge, in a digraph if and only if the digraph is Eulerian — see, for example, Corollary 6.1 of Gibbons [9]. Moreover, there are simple and efficient algorithms for finding Eulerian circuits — see for example Gibbons [9, Figure 6.5].

It is straightforward to verify that there is a correspondence between a $k$-ary de Bruijn sequence of order $n$ and a directed Eulerian circuit in $B_k(n-1)$, in which consecutive edges in the circuit correspond to consecutive $n$-tuples appearing in the sequence.

**Definition 1.3** ([2]). A $k$-ary $n$-window sequence $S = (s_i)$ is said to be an *orientable sequence of order $n$*, an $\mathcal{OS}_k(n)$, if $\mathbf{s}_n(i) \ne \mathbf{s}_n(j)^R$, for any $i, j$.

## 1.3 This paper

The remainder of this paper is structured as follows. We start, in Section 2, by developing new upper bounds on the period of an orientable sequence. In Section 3 we describe our novel general method of construction, which involves regarding $n$-tuples as edges in the de Bruijn graph of order $n - 1$. A first method of constructing a set of $n$-tuples with the desired properties is presented in Section 4. A second method, involving use of the Lempel homomorphism from the de Bruijn graph of order $n$ to the de Bruijn graph of

order $n-1$, is given in Section 5. For both constructions simple examples are given, and the periods of the orientable sequences generated are tabulated for small values of $n$ and $k$. This leads naturally to Section 6, where a table is given of the largest known periods for orientable sequences for $n$ and $k$ at most 8. As discussed briefly in the concluding section, Section 7, this table reveals that for $n > 3$ there remains a significant gap between the largest known period for an orientable sequence and the existing upper bound.

Preliminary versions of some of the results in this paper were presented at Sequences 2025 (London, February 2025) and the 5th Pythagorean Conference (Kalamata, June 2025).

## 2 New upper bounds on the period of an orientable sequence

### 2.1 Preliminaries

We start by giving some simple definitions and elementary results relating to these definitions.

**Definition 2.1.** Suppose $n \geq 1$ and $k \geq 2$. If $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ is a $k$-ary $n$-tuple, then $\mathbf{a}$ is said to be *uniform* if and only if $a_i = a_j$ for every $i, j \in \{0, 1, \ldots, n-1\}$.

So a uniform $n$-tuple appearing in a sequence corresponds to a string $a, a, \ldots, a$ of $n$ symbols $a$, which can be written as $a^n$.

**Definition 2.2.** Suppose $n \geq 1$ and $k \geq 2$. If $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ is a $k$-ary $n$-tuple, then $\mathbf{a}$ is said to be *alternating* if and only if there exist $c_0$ and $c_1$ such that $a_{2i} = c_0$ and $a_{2i+1} = c_1$ for every $i$ such that $i \geq 0$ and $2i + 1 \leq n - 1$ and $\mathbf{a}$ is not uniform.

So an alternating $n$-tuple appearing in a sequence corresponds to a string of the form $ab \ldots aba$ if $n$ is odd or $ab \ldots ab$ if $n$ is even, where $a$ and $b$ are distinct symbols, which can be written as $(ab)^{(n-1)/2}a$ or $(ab)^{(n/2)}$ respectively.

**Definition 2.3.** Suppose $n \geq 1$ and $k \geq 2$. If $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ is a $k$-ary $n$-tuple, then $\mathbf{a}$ is said to be *symmetric* if and only if $a_i = a_{n-1-i}$ for every $i \in \{0, 1, \ldots, n-1\}$. An $n$-tuple which is not symmetric is referred to as *asymmetric*.

So a symmetric $n$-tuple appearing in a sequence corresponds to a palindromic string of $n$ symbols. In what follows we write $\mathbf{p}$ for a palindrome where its length may be determined by context.

An Orientable sequence of order $n$ cannot contain palindromes of length $n$. This means that symmetric $n$-tuples do not appear in an $OS_k(n)$. This

property is reflected in the representation of the sequence in the de Bruijn digraph and we make use of this in what follows. We define two key subgraphs of the de Bruijn digraph.

**Definition 2.4.** Suppose $k \geq 2$ and $n \geq 1$. Let $B_k^*(n)$ be the subgraph of the de Bruijn digraph $B_k(n)$ with all the edges corresponding to symmetric $(n+1)$-tuples removed.

*Remark* 2.1. Since there are $k^{\lceil (n+1)/2 \rceil}$ symmetric $(n+1)$-tuples, $B_k^*(n)$ contains $k^{n+1} - k^{\lceil (n+1)/2 \rceil}$ edges.

**Definition 2.5.** Suppose $k \geq 2$ and $n \geq 2$. Suppose $S$ is an $\mathcal{OS}_k(n)$. Let $B(S, n)$ be the subgraph of $B_k(n-1)$ with vertices the vertices of $B_k(n-1)$ and with edges corresponding to those $n$-tuples which appear in either $S$ or $S^R$. We refer to $B(S, n)$ as the sequence-subgraph.

The following simple lemma is key.

**Lemma 2.1.** *Suppose $k \geq 2$ and $n \geq 2$. Suppose $S$ is an $\mathcal{OS}_k(n)$ of period $m$. Then:*

   i) *$B(S, n)$ contains $2m$ edges;*

  ii) *every vertex of $B(S, n)$ has in-degree equal to its out-degree; and*

 iii) *$B(S, n)$ is a subgraph of $B_k^*(n-1)$.*

*Proof.*      i) Since $S$ has period $m$, there are a total of $2m$ $n$-tuples appearing in $S$ and $S^R$. They are all distinct since $S$ is an $\mathcal{OS}_k(n)$.

  ii) $S$ and $S^R$ correspond to edge-disjoint Eulerian circuits in $B(S, n)$, and the result follows.

 iii) This is immediate since if $S$ is an $\mathcal{OS}_k(n)$, then neither $S$ nor $S^R$ can contain any symmetric $n$-tuples.

$\square$

## 2.2  The approach

The bound we give below derives from a careful analysis of the sequence-subgraph $B(S, n)$ of an $\mathcal{OS}_k(n)$. In particular we consider the impact on the possible number of edges in this subgraph arising from the constraints we give in Sections 2.3 and 2.4 below. Any upper bound on the number of edges in $B(S, n)$ immediately gives rise to a bound on the period of an orientable sequence, since the period is at most half the maximum cardinality of $B(S, n)$ (by Lemma 2.1(i)). In turn $|B(S, n)|$ is bounded above by the number of edges in $B_k^*(n-1)$, i.e. $k^n - k^{\lceil n/2 \rceil}$ by Lemma 2.11(iv), less the number of edges in $B_k^*(n-1)$ that cannot occur in $B(S, n)$.

In Sections 2.3 and 2.4 we describe two main ways in which we can show that certain edges in $B_k^*(n-1)$ cannot occur in $B(S,n)$ when $S$ is an $\mathcal{OS}_k(n)$ (in Corollaries 2.5 and 2.9). In particular we show that in two cases incoming edges to certain categories of vertex cannot occur, and also in two cases that outgoing edges from certain categories of vertex cannot occur.

This strategy is complicated by the fact that, as noted in Section 2.5, there is a danger of 'double counting' certain excluded edges. For example, Corollary 2.5 asserts that certain outgoing edges from a left-semi-symmetric $(n-1)$-tuple cannot occur, and Corollary 2.9 asserts that certain edges incoming to a uniform or symmetric $(n-1)$ tuple cannot occur. These two sets of excluded edges may overlap, and hence we need to take this into account; Lemma 2.10 is of key importance in this respect.

## 2.3 In-out-degree constraints on the sequence-subgraph

In this section and the next we consider properties of the sequence-subgraph for $S$ that can be derived from the assumption that $S$ is an $\mathcal{OS}_k(n)$. We first need the following definition.

**Definition 2.6.** Suppose $k \geq 2$ and $n \geq 3$. An $n$-tuple $(a_0, a_1, \ldots, a_{n-1})$ is said to be *left-semi-symmetric* if $a_i = a_{n-i-2}$, $0 \leq i \leq n-2$. Equivalently, $(a_0, a_1, \ldots, a_{n-1})$ is left-semi-symmetric if and only if $(a_0, a_1, \ldots, a_{n-2})$ is symmetric.

Analogously, an $n$-tuple $(a_0, a_1, \ldots, a_{n-1})$ is said to be *right-semi-symmetric* if $a_i = a_{n-i}$, $1 \leq i \leq n-1$. Equivalently, $(a_0, a_1, \ldots, a_{n-1})$ is right-semi-symmetric if and only if $(a_1, a_2, \ldots, a_{n-1})$ is symmetric.

So a left-semi-symmetric $n$-tuple corresponds to a string $\mathbf{p}a$ where $\mathbf{p}$ is a palindromic string of length $n-1$ and $a$ any symbol, and analogously a right-semi-symmetric $n$-tuple corresponds to a string $a\mathbf{p}$.

The fact that $a\mathbf{p}a$ is palindromic gives the following lemma.

**Lemma 2.2.** *Suppose $k \geq 2$ and $n \geq 3$. A vertex in $B_k^*(n)$ has:*

i) *in-degree $k-1$ if and only if it its label is left-semi-symmetric; otherwise it has in-degree $k$;*

ii) *out-degree $k-1$ if and only if it its label is right-semi-symmetric; otherwise it has out-degree $k$.*

*Proof.* For (i), the in-degree of every vertex in $B_k(n)$ is $k$. However, if (and only if) an inbound edge corresponds to a symmetric $(n+1)$-tuple, then this edge will not be in $B_k^*(n)$. Such an event can occur if and only if the vertex is labelled with a left-semi-symmetric $n$-tuple, and there can only be one such symmetric inbound edge. The result follows. The proof of (ii) follows using an exactly analogous argument. $\square$

This immediately tells us that some edges in $B_k^*(n-1)$ cannot occur in $B(S,n)$ if $S$ is a $\mathcal{OS}_k(n)$. However, before describing exactly when this occurs, we first need the following simple result which describes when a string is simultaneously of the form $\mathbf{p}a$ and of the form $a\mathbf{p}$.

**Lemma 2.3.** *Suppose $k \geq 2$ and $n \geq 4$. Suppose the $(n-1)$-tuple $(a_0, a_1, \ldots, a_{n-2})$ is both left-semi-symmetric and right-semi-symmetric. Then*

  *i) if $n$ is even then $(a_0, a_1, \ldots, a_{n-2})$ is uniform;*

  *ii) if $n$ is odd then $(a_0, a_1, \ldots, a_{n-2})$ is either uniform or alternating.*

*Proof.* Suppose the $(n-1)$-tuple $(a_0, a_1, \ldots, a_{n-2})$ is both left-semi-symmetric and right-semi-symmetric. Then $a_i = a_{n-i-3}$, $0 \leq i \leq n-3$, and $a_i = a_{n-i-1}$, $1 \leq i \leq n-2$. Since $n \geq 4$ this implies that there exist constants $c_0$ and $c_1$ such that $c_0 = a_{2i}$, $0 \leq 2i \leq n-2$, and $c_1 = a_{2j+1}$, $0 \leq 2j+1 \leq n-2$. Hence (ii) follows.

If $n$ is even then we have $a_{(n-2)/2} = a_{(n-2)/2-1}$, and hence $c_0 = c_1$ and (i) follows. □

The following result follows immediately from Lemmas 2.2 and 2.3.

**Corollary 2.4.** *Suppose $k \geq 2$ and $n \geq 4$. Consider a vertex in $B_k^*(n-1)$ with label $\mathbf{a} = (a_0, a_1, \ldots, a_{n-2})$, where $\mathbf{a}$ is non-uniform.*

  *i) if $n$ is even and $\mathbf{a}$ is left-semi-symmetric (or right-semi-symmetric), then its in-degree (resp. out-degree) is $k-1$ and its out-degree (resp. in-degree) is $k$;*

  *ii) if $n$ is odd and $\mathbf{a}$ is left-semi-symmetric (or right-semi-symmetric) and non-alternating, then its in-degree (resp. out-degree) is $k-1$ and its out-degree (resp. in-degree) is $k$.*

The above corollary immediately tells us that certain edges in $B_k^*(n-1)$ cannot occur in $B(S,n)$, as follows.

**Corollary 2.5.** *Suppose $k \geq 2$ and $n \geq 4$ and $S$ is an $\mathcal{OS}_k(n)$. Then*

  *i) if $n$ is even, for every vertex in $B_k^*(n-1)$ corresponding to a non-uniform left-semi-symmetric (or right-semi-symmetric) $(n-1)$-tuple $(a_0, a_1, \ldots, a_{n-2})$, there is an edge $(a_0, a_1, \ldots, a_{n-2}, x)$ in $B_k^*(n-1)$, for some $x$ (resp. an edge $(y, a_0, a_1, \ldots, a_{n-2})$ in $B_k^*(n-1)$, for some $y$), that is not in $B(S,n)$;*

  *ii) if $n$ is odd, for every vertex in $B_k^*(n-1)$ corresponding to a non-uniform non-alternating left-semi-symmetric (or right-semi-symmetric) $(n-1)$-tuple $(a_0, a_1, \ldots, a_{n-2})$, there is an edge $(a_0, a_1, \ldots, a_{n-2}, x)$ in $B_k^*(n-1)$ (resp. an edge $(y, a_0, a_1, \ldots, a_{n-2})$ in $B_k^*(n-1)$, for some $y$), for some $x$, that is not in $B(S,n)$.*

*Proof.* The result follows immediately from Lemma 2.1(ii) and Corollary 2.4. □

## 2.4 Degree-parity constraints on the sequence-subgraph

We first give the following simple lemma which shows that a string of the form $\mathbf{p}a$ or $a\mathbf{p}$ that is palindromic must be uniform.

**Lemma 2.6.** *Suppose $k \geq 2$ and $n \geq 2$. If an $n$-tuple is both left-semi-symmetric and symmetric then it is uniform. Similarly, if an $n$-tuple is both right-semi-symmetric and symmetric then it is uniform.*

*Proof.* Let $\mathbf{a} = (a_0, \ldots, a_{n-1})$ be an $n$-tuple which is both left-semi-symmetric and symmetric. Then $a_i = a_{n-2-i} = a_{n-1-i}$ for $i = 0, 1, \ldots, n-2$. It is immediate that $a_j = a_{j+1}$ for $j = 0, 1, \ldots, n-2$ and thus $\mathbf{a}$ is uniform. The second claim follows by an analogous argument. □

The following lemma is key.

**Lemma 2.7.** *Suppose $k \geq 2$ and $n \geq 2$. Suppose $S$ is an $\mathcal{OS}_k(n)$ and consider a vertex in $B(S, n)$ with label $\mathbf{a} = (a_0, a_1, \ldots, a_{n-2})$, where $\mathbf{a}$ is symmetric. Then $\mathbf{a}$ has even in-degree and even out-degree in $B(S, n)$.*

*Proof.* Both $S$ and $S^R$ correspond to an Eulerian circuit in $B(S, n)$, and these circuits are edge disjoint and cover all the edges of $B(S, n)$. If $\mathbf{a}$ is symmetric then both circuits pass through this vertex equally many times. It follows that $\mathbf{a}$ has even in-degree and even out-degree in $B(S, n)$. □

We also need the following.

**Lemma 2.8.** *Suppose $k \geq 2$ and $n \geq 2$, and consider a vertex in $B_k^*(n-1)$ with label $\mathbf{a} = (a_0, a_1, \ldots, a_{n-2})$. Then:*

  *i) if $k$ is even and $\mathbf{a}$ is uniform then the vertex has odd in-degree in $B_k^*(n-1)$ (and out-degree);*

  *ii) if $k$ is odd and $\mathbf{a}$ is symmetric and non-uniform then the vertex has odd in-degree in $B_k^*(n-1)$ (and out-degree).*

*Proof.*     i) Since $a$ is uniform it is also left-semi-symmetric and right-semi-symmetric, and hence by Lemma 2.2 it has in-degree and out-degree $k - 1$. Since $k$ is even the result follows.

  ii) Since $\mathbf{a}$ is symmetric and non-uniform it cannot be left-semi-symmetric or right-semi-symmetric by Lemma 2.6. Hence $\mathbf{a}$ has in-degree and out-degree $k$ and the result follows since $k$ is odd.

□

Combining Lemmas 2.7 and 2.8, and observing that $B(S, n)$ is necessarily a subgraph of $B_k^*(n-1)$, gives the following important result.

**Corollary 2.9.** *Suppose $k \geq 2$ and $n \geq 2$ and $S$ is an $\mathcal{OS}_k(n)$. Then*

*i) if $k$ is even, for every vertex in $B_k^*(n-1)$ corresponding to a uniform $(n-1)$-tuple $(a, a, ..., a)$, there are edges $(a, a, ..., a, x)$ and $(y, a, a, ..., a)$ in $B_k^*(n-1)$, for some $x, y$, that are not in $B(S, n)$.*

*ii) if $k$ is odd, for every vertex in $B_k^*(n-1)$ corresponding to a symmetric and non-uniform $(n-1)$-tuple $(a_0, a_1, \ldots, a_{n-2})$, there are edges $(a_0, a_1, \ldots, a_{n-2}, x)$ and $(y, a_0, a_1, \ldots, a_{n-2})$ in $B_k^*(n-1)$, for some $x$, that are not in $B(S, n)$.*

*Remark 2.2.* Observe that, for $n = 2$ and $k$ odd, Lemma 2.8 and Corollary 2.9 are trivially true since there are no symmetric non-uniform $(n-1)$-tuples.

## 2.5 Constraint interactions

The results above indicate in what circumstances certain edges in $B_k^*(n-1)$ cannot occur in $B(S, n)$ if $S$ is an $\mathcal{OS}_k(n)$. In particular we have considered cases where one of the incoming edges to a vertex corresponding to a symmetric or right-semi-symmetric $(n-1)$-tuple cannot occur in $B(S, n)$, and also where one of the outgoing edges from a vertex corresponding to a symmetric or left-semi-symmetric $(n-1)$-tuple cannot occur in $B(S, n)$. While we would like to add together the numbers of eliminated edges for each case, we need to ensure we avoid 'double counting'.

More specifically we need to consider when an edge can be outgoing from a symmetric or left-semi-symmetric $(n-1)$-tuple and also incoming to a symmetric or right-semi-symmetric $(n-1)$-tuple. This motivates the following result which captures when an $n$-tuple corresponds to:

(i) a string simultaneously of the form $\mathbf{p}a_{n-1}$ and of the form $a_0\mathbf{p}'$ (i.e. a string of form $a^n$ or $(ab)^{n/2}$);

(ii) a string simultaneously of the form $\mathbf{p}a_{n-1}$ and of the form $a_0 a_1 \mathbf{p}'$ (i.e. a string of form $(aab)^{n/3}$ or $(aba)^{(n-1)/3}a$ or $(abb)^{(n-2)/3}ab$);

(iii) a string simultaneously of the form $\mathbf{p}a_{n-2}a_{n-1}$ and of the form $a_0\mathbf{p}'$ (i.e. a string of the form $(abb)^{n/3}$ or $(aab)^{(n-1)/3}a$ or $(aba)^{(n-2)/3}ab$);

(iv) a string simultaneously of the form $\mathbf{p}a_{n-2}a_{n-1}$ and of the form $a_0 a_1 \mathbf{p}'$ (i.e. a string of the form $(abba)^{n/4}$ or $(abac)^{(n-1)/4}a$ or $(abba)^{(n-2)/4}ab$ or $(abcb)^{(n-3)/4}abc$).

**Lemma 2.10.** *Suppose $k \geq 2$. Suppose $\mathbf{a} = (a_0, a_1, \ldots, a_{n-2})$ and $\mathbf{b} = (a_1, a_2, \ldots, a_{n-1})$ are $k$-ary $(n-1)$-tuples, connected by the edge $(a_0, a_1, \ldots, a_{n-1})$.*

*i) If $n \geq 3$, $\mathbf{a}$ is symmetric and $\mathbf{b}$ is symmetric then either $\mathbf{a}$ and $\mathbf{b}$ are both uniform or $n$ is even and $\mathbf{a}$ and $\mathbf{b}$ are both alternating.*

*ii)* If $n \geq 5$, $\mathbf{a}$ *is symmetric and* $\mathbf{b}$ *is right-semi-symmetric then there exist* $c_j$, $0 \leq j \leq 2$, *such that* $a_{3i+j} = c_j$, *for every* $i$ *and* $0 \leq j \leq 2$. *Moreover if* $n \equiv 0 \pmod 3$ *then* $c_0 = c_1$, *if* $n \equiv 1 \pmod 3$ *then* $c_0 = c_2$, *and if* $n \equiv 2 \pmod 3$ *then* $c_1 = c_2$.

*iii)* If $n \geq 5$, $\mathbf{a}$ *is left-semi-symmetric and* $\mathbf{b}$ *is symmetric then there exist* $c_j$, $0 \leq j \leq 2$, *such that* $a_{3i+j} = c_j$, *for every* $i$ *and* $0 \leq j \leq 2$. *Moreover if* $n \equiv 0 \pmod 3$ *then* $c_1 = c_2$, *if* $n \equiv 1 \pmod 3$ *then* $c_0 = c_1$, *and if* $n \equiv 2 \pmod 3$ *then* $c_0 = c_2$.

*iv)* If $n \geq 5$, $\mathbf{a}$ *is left-semi-symmetric and* $\mathbf{b}$ *is right-semi-symmetric then there exist* $c_j$, $0 \leq j \leq 3$, *such that* $a_{4i+j} = c_j$, *for every* $i$ *and* $0 \leq j \leq 3$. *Moreover if* $n \equiv 0 \pmod 4$ *then* $c_0 = c_1$ *and* $c_2 = c_3$, *if* $n \equiv 1 \pmod 4$ *then* $c_0 = c_2$, *if* $n \equiv 2 \pmod 4$ *then* $c_0 = c_3$ *and* $c_1 = c_2$, *and if* $n \equiv 3 \pmod 4$ *then* $c_1 = c_3$.

*Proof.* i) By symmetry of $\mathbf{a}$ and $\mathbf{b}$, respectively, $a_i = a_{n-2-i}$ and $a_{i+1} = a_{n-1-i}$ for every $i$, $0 \leq i \leq n-2$. Hence there exist $c_0$ and $c_1$ such that $a_{2i} = c_0$ and $a_{2i+1} = c_1$ for every $i$. If $n$ is odd then $c_0 = c_1$ and the result follows.

ii) By symmetry of $\mathbf{a}$, $a_i = a_{n-2-i}$ for every $i$ ($0 \leq i \leq n-2$), and by right-semi-symmetry of $\mathbf{b}$, $a_j = a_{n+1-j}$ for every $j$ ($2 \leq j \leq n-1$). Hence $a_i = a_{i+3}$, $0 \leq i < n-3$. Now, since $n \geq 5$, $a_{3i+j} = c_j$, for every $i$ and $0 \leq j \leq 2$ for some $c_j$. From symmetry of $\mathbf{a}$, we have $c_0 = a_0 = a_{n-2}$; if $n \equiv 0 \pmod 3$ then $a_{n-2} = c_1$, i.e. $c_0 = c_1$. From right-semi-symmetry of $\mathbf{b}$, $c_2 = a_2 = a_{n-1}$; if $n \equiv 1 \pmod 3$ then $a_{n-1} = c_0$, and so $c_0 = c_2$, and finally if $n \equiv 2 \pmod 3$ then $a_{n-1} = c_1$, i.e. $c_1 = c_2$.

iii) By left-semi-symmetry of $\mathbf{a}$, $a_i = a_{n-3-i}$ for every $i$ ($0 \leq i \leq n-3$), and by symmetry of $\mathbf{b}$, $a_j = a_{n-j}$ for every $j$ ($1 \leq j \leq n-1$). Using exactly the same argument as for (ii), $a_{3i+j} = c_j$, for every $i$ and $0 \leq j \leq 2$ for some $c_j$. By the symmetry of $\mathbf{b}$, $c_1 = a_1 = a_{n-1}$; if $n \equiv 0 \pmod 3$ then $a_{n-1} = c_2$ (and so $c_1 = c_2$), and if $n \equiv 1 \pmod 3$ then $a_{n-1} = c_0$ and so $c_1 = c_0$. Finally, by left-semi-symmetry of $\mathbf{a}$, $c_0 = a_0 = a_{n-3}$, and hence if $n \equiv 2 \pmod 3$ then $a_{n-3} = c_2$ and so $c_0 = c_2$.

iv) By left-semi-symmetry of $\mathbf{a}$, $a_i = a_{n-3-i}$ for every $i$ ($0 \leq i \leq n-3$) and by right-semi-symmetry of $\mathbf{b}$, $a_j = a_{n+1-j}$ for every $j$ ($2 \leq j \leq n-1$). Hence $a_i = a_{i+4}$, $0 \leq i \leq n-3$. Thus, since $n \geq 5$, $a_{4i+j} = c_j$, for every $i$ and $0 \leq j \leq 3$ for some $c_j$. From left-semi-symmetry of $\mathbf{a}$, $c_0 = a_0 = a_{n-3}$; if $n \equiv 0 \pmod 4$ then $a_{n-3} = c_1$ and so $c_0 = c_1$, if $n \equiv 1 \pmod 4$ then $a_{n-3} = c_2$ and so $c_0 = c_2$, and if $n \equiv 2 \pmod 4$ then $a_{n-3} = c_3$ and so $c_0 = c_3$. Further, we have

$c_1 = a_1 = a_{n-4}$; if $n \equiv 3 \pmod 4$ then $a_{n-4} = c_3$ and so $c_1 = c_3$. From right-semi-symmetry of $\mathbf{b}$, we have $c_2 = a_2 = a_{n-1}$; if $n \equiv 0 \pmod 4$ then $a_{n-1} = c_3$ (and so $c_2 = c_3$), and finally if $n \equiv 2 \pmod 4$ then $a_{n-1} = c_1$, i.e. $c_1 = c_2$.

$\square$

## 2.6 Counting special types of $n$-tuple

The following simple enumerative result will be of use below. We omit the proof as it is elementary.

**Lemma 2.11.** *Suppose $n \geq 3$ and $k \geq 2$. Then*

  i) *the number of symmetric $k$-ary $n$-tuples is $k^{\lceil n/2 \rceil}$;*

 ii) *the number of uniform $k$-ary $n$-tuples is $k$.*

iii) *the number of symmetric non-uniform $k$-ary $n$-tuples is $k^{\lceil n/2 \rceil} - k$;*

 iv) *the number of asymmetric $k$-ary $n$-tuples is $k^n - k^{\lceil n/2 \rceil}$;*

  v) *the number of alternating $k$-ary $n$-tuples is $k(k-1)$;*

 vi) *the number of non-uniform non-alternating symmetric $k$-ary $n$-tuples is $k^{(n+1)/2} - k^2$ if $n$ is odd, and $k^{n/2} - k$ if $n$ is even;*

vii) *the number of left-semi-symmetric $k$-ary $n$-tuples is $k^{\lceil (n+1)/2 \rceil}$;*

viii) *the number of non-uniform left-semi-symmetric $k$-ary $n$-tuples is $k^{\lceil (n+1)/2 \rceil} - k$;*

 ix) *the number of non-uniform non-alternating left-semi-symmetric $k$-ary $n$-tuples is $k^{(n+2)/2} - k^2$ if $n$ is even, and $k^{(n+1)/2} - k$ if $n$ is odd.*

## 2.7 Developing the bound

The following notation is intended to simplify the arguments. Let $U_{\text{in}}$ and $U_{\text{out}}$ be the sets of incoming and outgoing edges excluded by Corollary 2.5, and $P_{\text{in}}$ and $P_{\text{out}}$ be the sets of incoming and outgoing edges excluded by Corollary 2.9.

$U_{\text{in}}$ contains edges corresponding to strings of the following forms:

 (i) $ya\mathbf{p}$ where $\mathbf{p}$ is palindromic but $a\mathbf{p}$ is not uniform, $n$ is even;

(ii) $ya\mathbf{p}$ where $\mathbf{p}$ is palindromic but $a\mathbf{p}$ is not of the form $(ab)^{(n-1)/2}a$, n odd.

$U_{\text{out}}$ contains edges corresponding to strings of the following forms:

 (i) $\mathbf{p}ax$ where $\mathbf{p}$ is palindromic but $\mathbf{p}a$ is not uniform, $n$ is even;

(ii) $\mathbf{p}ax$ where $\mathbf{p}$ is palindromic but $\mathbf{p}a$ is not of the form $(ab)^{(n-1)/2}a$, $n$ is odd.

$P_{\mathrm{in}}$ contains edges corresponding to strings of the following forms:

(i) $ya^{n-1}$, $k$ is even;

(ii) $y\mathbf{p}$ where $\mathbf{p}$ is palindromic but not uniform, $k$ is odd.

$P_{\mathrm{out}}$ contains edges corresponding to strings of the following forms:

(i) $a^{n-1}x$, $k$ is even;

(ii) $\mathbf{p}x$ where $\mathbf{p}$ is palindromic but not uniform, $k$ is odd.

The discussion at the beginning of the section leads to the following key lemma.

**Lemma 2.12.** *If $k \geq 2$ and $S$ is an $\mathcal{OS}_k(n)$ then:*

$$
|B(S,n)| \leq \begin{cases}
k^2 - k - |P_{out}|, & \text{if } n = 2 \\
k^3 - k^2 - |P_{out}| - |P_{in}| + |P_{out} \cap P_{in}|, & \text{if } n = 3 \\
k^4 - k^2 - |U_{out}| - |P_{out}|, & \text{if } n = 4 \\
k^n - k^{\lceil n/2 \rceil} - |U_{out}| - |U_{in}| - |P_{out}| - |P_{in}| \\
\quad + |U_{out} \cap P_{in}| + |P_{out} \cap U_{in}| \\
\quad + |U_{out} \cap U_{in}| + |P_{out} \cap P_{in}|, & \text{if } n > 4.
\end{cases}
$$

*where $|X \cap Y|$ denotes the maximum possible cardinality for such a set.*

*Proof.* The argument for $n = 2$ is immediate. A similar comment applies when $n = 3$. The $n = 4$ and $n \geq 5$ cases follow from observing that $U_{\mathrm{in}} \cap P_{\mathrm{in}} = U_{\mathrm{out}} \cap P_{\mathrm{out}} = \emptyset$ because a symmetric $(n-1)$-tuple can neither be non-uniform left-semi-symmetric nor non-uniform right-semi-symmetric, from Lemma 2.6. □

*Remark* 2.3. The reason to restrict the sets considered when $n \leq 4$ is because certain sets are empty or might be equal for small $n$. Moreover Corollary 2.5 only applies for $n \geq 4$.

## 2.8 The bound

We can now give the bound.

**Theorem 2.13.** *Suppose $k \geq 2$ and $m$ is the period of an $\mathcal{OS}_k(n)$.*

*i) If $n = 2$ then:*

$$
m \leq \begin{cases}
\frac{k^2 - k}{2}, & \text{if } k \text{ is odd} \\
\frac{k^2 - 2k}{2}, & \text{if } k \text{ is even}
\end{cases}
$$

*ii) If $n = 3$ then:*

$$m \leq \begin{cases} \frac{k^3 - k^2}{2}, & \text{if } k \text{ is odd} \\ \frac{k^3 - k^2 - 2k}{2}, & \text{if } k \text{ is even} \end{cases}$$

*iii) If $n = 4$ then:*

$$m \leq \begin{cases} \frac{k^4 - 3k^2 + 2k}{2}, & \text{if } k \text{ is odd} \\ \frac{k^4 - 2k^2}{2}, & \text{if } k \text{ is even} \end{cases}$$

*iv) If $n > 4$ then:*

$$m \leq \begin{cases} \frac{k^n - 3k^{(n+1)/2} - 2k^{(n-1)/2} + k^3 + 3k^2}{2}, & \text{if } n \text{ and } k \text{ are odd} \\ \frac{k^n - 3k^{(n+1)/2} + k^3 + k^2 - 2k}{2}, & \text{if } n \text{ is odd and } k \text{ is even} \\ \frac{k^n - 5k^{n/2} + 4k^2}{2}, & \text{if } n \text{ is even and } k \text{ is odd} \\ \frac{k^n - 3k^{n/2} + k^2 - k}{2}, & \text{if } n \text{ and } k \text{ are even} \end{cases}$$

*Proof.* The proof builds on, and uses the notation of, Lemma 2.12.

i) $n = 2$. If $k$ is odd then $P_{\text{out}} = \emptyset$ by Corollary 2.9(ii), since there are no symmetric non-uniform 1-tuples. If $k$ is even then, by Corollary 2.9(i), $|P_{\text{out}}| = k$, since there are $k$ uniform 1-tuples by Lemma 2.11(ii). The result follows from Lemma 2.12.

ii) $n = 3$. If $k$ is odd then, by Corollary 2.9(ii), $|P_{\text{in}}| = |P_{\text{out}}|$ equals the number of symmetric non-uniform 2-tuples, i.e. zero by Lemma 2.11(iii).

If $k$ is even then, by Corollary 2.9(i), $|P_{\text{in}}| = |P_{\text{out}}|$ equals the number of uniform 2-tuples, i.e. $k$ by Lemma 2.11(ii). Moreover, $P_{\text{in}} \cap P_{\text{out}} = \emptyset$, since an edge can only be outgoing from a uniform $(n-1)$-tuple and incoming to a uniform $(n-1)$-tuple if it corresponds to a uniform $n$-tuple, and such an edge cannot occur in $B_k^*(n-1)$. The result follows from Lemma 2.12.

iii) $n = 4$. By Corollary 2.5(i), $|U_{\text{out}}|$ equals the number of non-uniform left-semi-symmetric 3-tuples, i.e. $k^2 - k$ by Lemma 2.11(viii). If $k$ is odd then, by Corollary 2.9(ii), $|P_{\text{out}}|$ equals the number of symmetric non-uniform 3-tuples, i.e. $k^2 - k$ by Lemma 2.11(iii). If $k$ is even then, by Corollary 2.9(i), $|P_{\text{out}}|$ equals the number of uniform 3-tuples, i.e. $k$ by Lemma 2.11(ii). The result follows from Lemma 2.12.

iv) $n > 4$.

**Case A: $n$ odd**. By Corollary 2.5(ii), $|U_{\text{out}}| = |U_{\text{in}}|$ equals the number of non-uniform non-alternating left-semi-symmetric $(n-1)$-tuples, i.e. $k^{(n+1)/2} - k^2$ by Lemma 2.11(ix). Also, by Lemma 2.10(iv), $|U_{\text{out}} \cap U_{\text{in}}| = k^3 - k^2$ since if, in the statement of the lemma, we choose $c_1 \neq c_3$ if $n \equiv 1 \pmod 4$, and $c_0 \neq c_2$ if $n \equiv 3 \pmod 4$, then $\mathbf{a}$ and $\mathbf{b}$

are non-uniform and non-alternating and there are $k^2(k-1) = k^3 - k^2$ possibilities for the values $c_i$.

If $k$ is odd then, by Corollary 2.9 (ii), $|P_{\text{out}}| = |P_{\text{in}}|$ equals the number of symmetric non-uniform $(n-1)$-tuples, i.e. $k^{(n-1)/2} - k$ by Lemma 2.11(iii).

By Lemma 2.10(ii),(iii), $|U_{\text{out}} \cap P_{\text{in}}| = |P_{\text{out}} \cap U_{\text{in}}| = k(k-1)$ (removing the case where the $c_i$ are all equal).

By Lemma 2.10(i), $P_{\text{out}} \cap P_{\text{in}} = \emptyset$.

If $k$ is even then, by Corollary 2.9(i), $|P_{\text{out}}| = |P_{\text{in}}|$ equals the number of uniform $(n-1)$-tuples, i.e. $k$ by Lemma 2.11(ii).

By Corollary 2.9(i), $P_{\text{out}}$ and $P_{\text{in}}$ only contain edges out-going/in-going from/to uniform $(n-1)$-tuples, and by Corollary 2.5(ii) $U_{\text{out}}$ and $U_{\text{in}}$ only contain edges that are out-going/in-going from/to non-uniform $(n-1)$-tuples, and hence $|U_{\text{out}} \cap P_{\text{in}}| = |P_{\text{out}} \cap U_{\text{in}}| = \emptyset$.

By Corollary 2.9(i), $P_{\text{out}}$ and $P_{\text{in}}$ only contain edges out-going/in-going from/to uniform $(n-1)$-tuples, and hence an edge in $P_{\text{out}} \cap P_{\text{in}}$ must be uniform, i.e. $P_{\text{out}} \cap P_{\text{in}} = \emptyset$.

The result follows from Lemma 2.12.

**Case B:** $n$ **even**. By Corollary 2.5(i), $|U_{\text{out}}| = |U_{\text{in}}|$ equals the number of non-uniform left-semi-symmetric $(n-1)$-tuples, i.e. $k^{n/2} - k$ by Lemma 2.11(viii). Additionally, by Lemma 2.10(iv), $|U_{\text{out}} \cap U_{\text{in}}| = k(k-1)$, by choosing $c_0$ and $c_2$ to be distinct.

If $k$ is odd then, by Corollary 2.9(ii), $|P_{\text{out}}| = |P_{\text{in}}|$ equals the number of symmetric non-uniform $(n-1)$-tuples, i.e. $k^{n/2} - k$ by Lemma 2.11(iii).

By Lemma 2.10(ii),(iii), $|U_{\text{out}} \cap P_{\text{in}}| = |P_{\text{out}} \cap U_{\text{in}}| = k(k-1)$, by ensuring that $c_0$, $c_1$ and $c_2$ are not all equal.

By Lemma 2.10(i), $|P_{\text{out}} \cap P_{\text{in}}| = k(k-1)$, i.e. the number of non-uniform alternating $(n-1)$-tuples.

If $k$ is even then, by Corollary 2.9(i), $|P_{\text{out}} = |P_{\text{in}}|$ equals the number of uniform $(n-1)$-tuples, i.e. $k$ by Lemma 2.11(i).

By Corollary 2.9(i), $P_{\text{out}}$ and $P_{\text{in}}$ only contain edges out-going/in-going from/to uniform $(n-1)$-tuples, and by Corollary 2.5(i) $U_{\text{out}}$ and $U_{\text{in}}$ only contain edges that are out-going/in-going from/to non-uniform $(n-1)$-tuples, and hence $|U_{\text{out}} \cap P_{\text{in}}| = |P_{\text{out}} \cap U_{\text{in}}| = \emptyset$.

Finally, it is immediate that $P_{\text{out}} \cap P_{\text{in}} = \emptyset$.

The result follows from Lemma 2.12.

$\square$

## 2.9 Numerical results

The bounds resulting from Theorem 2.13 are tabulated for small $k$ and $n$ in Table 1. The numbers given in brackets are the bounds derived from [2, Theorem 4.11], and are provided for comparison purposes.

Table 1: Bounds — new and (old) — on the period of an $\mathcal{OS}_k(n)$

| $n$ | $k = 2$ | $k = 3$ | $k = 4$ | $k = 5$ | $k = 6$ | $k = 7$ | $k = 8$ |
|---|---|---|---|---|---|---|---|
| 2 | 0 | 3 | 4 | 10 | 12 | 21 | 24 |
|   | (0) | (3) | (4) | (10) | (12) | (21) | (24) |
| 3 | 0 | 9 | 20 | 50 | 84 | 147 | 216 |
|   | (1) | (9) | (22) | (50) | (87) | (147) | (220) |
| 4 | 4 | 30 | 112 | 280 | 612 | 1134 | 1984 |
|   | (5) | (33) | (118) | (290) | (627) | (1155) | (2012) |
| 5 | 8 | 99 | 452 | 1450 | 3684 | 8085 | 15896 |
|   | (11) | (105) | (478) | (1490) | (3777) | (8211) | (16124) |
| 6 | 21 | 315 | 1958 | 7550 | 23019 | 58065 | 130332 |
|   | (27) | (336) | (2014) | (7680) | (23217) | (58464) | (130812) |
| 7 | 44 | 972 | 7844 | 38100 | 138144 | 408072 | 1042712 |
|   | (55) | (1032) | (8062) | (38640) | (139317) | (410256) | (1046524) |
| 8 | 105 | 3096 | 32390 | 193800 | 837879 | 2876496 | 8382492 |
|   | (119) | (3189) | (32638) | (194630) | (839157) | (2879835) | (8386556) |
| 9 | 212 | 9423 | 129572 | 971350 | 5027304 | 20149437 | 67059992 |
|   | (239) | (9645) | (130558) | (974390) | (5034957) | (20166027) | (67092476) |

# 3 Eulerian cycles in the de Bruijn digraph

## 3.1 Overview

Inspired by the correspondence between $k$-ary de Bruijn sequences of order $n$ and Eulerian cycles in the order $n-1$ de Bruijn graph $B_k(n-1)$ (see Section 1.2), we next describe a correspondence between orientable sequences of order $n$ and Eulerian cycles in subgraphs of $B_k(n-1)$, where these subgraphs satisfy certain special properties. If we can then construct subgraphs with these special properties which admit Eulerian cycles then we have a simple method of generating orientable sequences. Note that the graphs we are concerned with here are an example of Rauzy Graphs [14].

**Definition 3.1.** Suppose $T$ is a subgraph of the de Bruijn digraph $B_k(n)$ for some $n \geq 2$ and $k \geq 2$. $T$ is said to be *antisymmetric* if the following property holds.

Suppose $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1}), \mathbf{y} = (y_0, y_1, \ldots, y_{n-1}) \in \mathbb{Z}_k^n$, i.e. they are vertices in $T$. Then if $(\mathbf{x}, \mathbf{y})$ is an edge in $T$, then $(\mathbf{y}^R, \mathbf{x}^R)$ is not an edge in $T$.

**Definition 3.2.** Suppose $S$ is a periodic $k$-ary $n$-window sequence of period $m$ for some $n \geq 2$, $k \geq 2$ and $m \geq 1$. Then the *edge-graph* $E_n(S)$ of $S$ is

defined to be the subgraph of $B_k(n-1)$ whose directed edges correspond to $n$-tuples appearing in a period of $S$, i.e. with edge set

$$\{\mathbf{s}_n(i) \ : \ 0 \leq i \leq m-1\}$$

and whose vertices are those vertices of $B_k(n-1)$ that have in-degree at least one.

We can now state a key lemma.

**Lemma 3.1.** *Suppose $S$ is a $k$-ary periodic $n$-window sequence. Then $S$ is an $\mathcal{OS}_k(n)$ if and only if $E_n(S)$ is antisymmetric in $B_k(n-1)$.*

*Proof.* Suppose $S$ is an $\mathcal{OS}_k(n)$. Then, by definition, $\mathbf{s}_n(i) \neq \mathbf{s}_n(j)^R$, for any $i, j$. Hence, again by definition, this means that $\mathbf{a} \neq \mathbf{b}^R$ for any edges $\mathbf{a}, \mathbf{b}$ in $E_n(S)$. Hence $E_n(S)$ is antisymmetric.

Now suppose $E_n(S)$ is antisymmetric, and hence $\mathbf{a} \neq \mathbf{b}^R$ for any edges $\mathbf{a}, \mathbf{b}$ in $E_n(S)$. Thus, $\mathbf{s}_n(i) \neq \mathbf{s}_n(j)^R$, for any $i, j$, and so $S$ is an $\mathcal{OS}_k(n)$. $\square$

This enables us to give our main result.

**Theorem 3.2.** *If $S$ is an $\mathcal{OS}_k(n)$ of period $m$ then $E_n(S)$ is an antisymmetric Eulerian subgraph of $B_k(n-1)$ containing $m$ edges. Moreover, if $T$ is an antisymmetric Eulerian subgraph of $B_k(n-1)$ with $m$ edges, then there exists an $\mathcal{OS}_k(n)$ $S$ of period $m$ such that $E_n(S) = T$.*

*Proof.* Suppose $S = (s_i)$ is an $\mathcal{OS}_k(n)$ of period $m$. Then, by Lemma 3.1, $E_n(S)$ is antisymmetric in $B_k(n-1)$. Also, trivially, $E_n(S)$ contains $m$ edges. Finally, if $(x_1, x_2, \ldots, x_{n-1})$ is a vertex in $E_n(S)$, then the incoming edges all have labels of the form $(x, x_1, x_2, \ldots, x_{n-1})$ for some $x$, and correspond to $\mathbf{s}_n(i)$ for some $i$. Then the edge corresponding to $\mathbf{s}_n(i+1)$ will have a label of the form $(x_1, x_2, \ldots, x_{n-1}, y)$ for some $y$, and so for every incoming edge there is an outgoing edge, and vice versa. Moreover, by the definition of an orientable sequence, distinct values of $x$ correspond to distinct values of $y$. Hence the in-degree of every vertex is the same as the out-degree. Finally, note that $E_n(S)$ is connected since we only include vertices in $E_n(S)$ with in-degree greater than zero, and the sequence $S$ defines a path incorporating every vertex in $E_n(S)$.

Now suppose $T$ is an antisymmetric Eulerian subgraph of $B_k(n-1)$ with $m$ edges. Then there exists an Eulerian circuit of length $m$ in $B_k(n-1)$. This circuit corresponds to a sequence $S$ with period $m$ in the natural way. $S$ is clearly an $n$-window sequence since the circuit visits every edge exactly once, and each edge corresponds to a unique $n$-tuple. It is straightforward to verify that $E_n(S) = T$. Finally, $S$ is an $\mathcal{OS}_k(n)$ from Lemma 3.1. $\square$

*Remark* 3.1. In general, for any antisymmetric Eulerian subgraph of $B_k(n-1)$ $T$, there will exist many orientable sequences with edge-graph $T$, since there may be many different Eulerian circuits in $T$, each corresponding to a different sequence.

## 3.2 Implications

Theorem 3.2 means that if we can construct an antisymmetric Eulerian subgraph of $B_k(n-1)$ with $m$ edges for 'large' $m$ (i.e. with $m$ close to the maximum possible), then we will immediately have a set of orientable sequences with period close to the maximum.

As a result, in the remainder of this paper we consider ways of constructing sets of edges in $B_k(n-1)$ that define an antisymmetric Eulerian subgraph of $B_k(n-1)$.

# 4 A simple construction

## 4.1 The subgraph

We next give a very simple way of establishing an antisymmetric Eulerian subgraph of $B_k(n-1)$.

**Definition 4.1.** If $n \geq 2$ and $k \geq 3$ let $A_k(n)$ be the subgraph of $B_k(n-1)$ with edges corresponding to the $n$-tuples $(a_0, a_1, \ldots, a_{n-1})$, where $a_i \in \mathbb{Z}_k$, for which $a_{n-1} - a_0 \in \{1, 2, \ldots \lfloor (k-1)/2 \rfloor\}$.

The following two results, the first of which is elementary, demonstrate the importance of $A_k(n)$.

**Lemma 4.1.** *Suppose $n \geq 2$ and $k \geq 3$. $A_k(n)$ is antisymmetric in the de Bruijn digraph $B_k(n-1)$ and has $k^{n-1} \left\lfloor \frac{k-1}{2} \right\rfloor$ edges.*

**Theorem 4.2.** *Suppose $n \geq 2$ and $k \geq 5$. $A_k(n)$ is an antisymmetric Eulerian subgraph of $B_k(n-1)$ with $k^{n-1} \left\lfloor \frac{k-1}{2} \right\rfloor$ edges.*

*Proof.* From Lemma 4.1 we need only show that $A_k(n)$ is connected and that every vertex has in-degree equal to its out-degree.

Consider any vertex $\mathbf{u} = (u_1, u_2, \ldots, u_{n-1})$ in $A_k(n)$. An incoming edge

$$(s, u_1, u_2, \ldots, u_{n-1})$$

must satisfy $u_{n-1} - s \in \{1, 2, \ldots \lfloor (k-1)/2 \rfloor\}$. Similarly an outgoing edge

$$(u_1, u_2, \ldots, u_{n-1}, t)$$

must satisfy $t - u_1 \in \{1, 2, \ldots \lfloor (k-1)/2 \rfloor\}$. Regardless of the values of $u_1$ and $u_{n-1}$ there are clearly $\lfloor (k-1)/2 \rfloor$ possible values for both $s$ and $t$, i.e. the in-degree of every vertex is the same as its out-degree; in both cases it will equal $\lfloor (k-1)/2 \rfloor$.

To show that $A_k(n)$ is connected observe that there will always exist a path from

$$(u_1, u_2, \ldots, u_{n-1}) \text{ to } (u_1, u_2, \ldots, u_{j-1}, u_j + 1, u_{j+1}, \ldots, u_{n-1})$$

for any $j$. This follows since, if $k \geq 5$, there is an edge from

$$(u_1, u_2, \ldots, u_{n-1}) \text{ to } (u_2, u_3, \ldots, u_{n-1}, u_1 + 1)$$

and also an edge from

$$(u_1, u_2, \ldots, u_{n-1}) \text{ to } (u_2, u_3, \ldots, u_{n-1}, u_1 + 2).$$

This enables a path of length $n$ to be constructed from $(u_1, u_2, \ldots, u_{n-1})$ to $(u_1+1, u_2+1, \ldots, u_{j-1}+1, u_j+2, u_{j+1}+1, \ldots, u_{n-1}+1)$ for any $j$, and then a path of length $2n$ to $(u_1+2, u_2+2, \ldots, u_{j-1}+2, u_j+3, u_{j+1}+2, \ldots, u_{n-1}+2)$, etc., to get a path to $(u_1+k, u_2+k, \ldots, u_{j-1}+k, u_j+k+1, u_{j+1}+k, \ldots, u_{n-1}+k)$ — which is equal to $(u_1, u_2, \ldots, u_{j-1}, u_j + 1, u_{j+1}, \ldots, u_{n-1})$.

Hence, inductively, there exists a path from $(u_1, u_2, \ldots, u_{n-1})$ to any vertex $(v_1, v_2, \ldots, v_{n-1})$.

The result follows. □

*Remark* 4.1. We need to assume $k \geq 5$ since, unfortunately, $A_k(n)$ is not connected if $k < 5$. If $k = 3$ and $n = 3$ we have:

$$00 \to 01 \to 11 \to 12 \to 22 \to 20 \to 00$$

and

$$02 \to 21 \to 10 \to 02$$

i.e. circuits of length 6 and 3. Similarly if $k = 3$ and $n = 4$ we obtain three circuits each of length 9. Similar problems arise for $k = 4$, because for $k = 3$ and $k = 4$ the in-degree and out-degree of every vertex in $A_n(k)$ is only 1.

Combining Theorems 4.2 and 3.2 we immediately obtain the following.

**Corollary 4.3.** *If $n \geq 2$ and $k \geq 5$ then there exists an $\mathcal{OS}_k(n)$ with period*

$$k^{n-1} \left\lfloor \frac{k-1}{2} \right\rfloor.$$

**Example 4.1.** As an example of Corollary 4.3, consider the case $k = 5$ and $n = 3$. The 50 3-tuples in $A_5(3)$ are listed in Table 2, and a period of an $\mathcal{OS}_5(3)$ containing these 50 3-tuples is:

[00123 40112 23344 00213 24304 21431 03142 03204 10224 41133].

Table 5 tabulates the periods of the generated sequences for small $k$ and $n$.

Comparison of the bound in Theorem 2.13 with Corollary 4.3 shows that the sequences have optimal period for $n = 2$ and $n = 3$ if $k$ is odd.

Table 2: 5-ary 3-tuples in $A_5(3)$

| 001 | 002 | 102 | 103 | 203 | 204 | 304 | 300 | 400 | 401 |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 011 | 012 | 112 | 113 | 213 | 214 | 314 | 310 | 410 | 411 |
| 021 | 022 | 122 | 123 | 223 | 224 | 324 | 320 | 420 | 421 |
| 031 | 032 | 132 | 133 | 233 | 234 | 334 | 330 | 430 | 431 |
| 041 | 042 | 142 | 143 | 243 | 244 | 344 | 340 | 440 | 441 |

## 4.2 Variants

There are a number of other antisymmetric Eulerian subgraphs of $B_k(n-1)$ with the same number of edges as $A_k(n)$. We briefly mention two such examples, where in both cases we assume $k \geq 5$ and $n \geq 2$.

- For odd $k$, define $C_k(n)$, where $(a_0, a_1, \ldots, a_{n-1})$ is an edge in $C_k(n)$ if and only if $a_{n-1} - a_0$ is odd. It is immediate to see that $C_k(n)$ is antisymmetric since if $(a_0, a_1, \ldots, a_{n-1})$ is an edge then $a_0 - a_{n-1}$ is even, since $k$ is odd. It can be shown that $C_k(n)$ is Eulerian using a similar argument to that for $A_n(k)$.

- A second variant constrains more than the first and last elements of an $n$-tuple. Suppose $t \leq n/2$. Let $A_k(n, t)$ be the subgraph of $B_k(n-1)$ with edges equal to the following set of $k$-ary $n$-tuples

$$\left\{ (a_0, a_1, \ldots, a_{n-1}) : \sum_{i=n-t}^{n-1} a_i - \sum_{i=0}^{t-1} a_i \in \{1, 2, \ldots \lfloor (k-1)/2 \rfloor\} \right\}.$$

  If the $n$-tuple $\mathbf{s}$ is an element of $A_k(n, t)$, then $\mathbf{s}^R \notin A_k(n, t)$, i.e. it is antisymmetric. It again follows that it is Eulerian using a similar argument to that for $A_k(n)$.

There are, no doubt, further variants that could be devised.

## 5 A construction using the Lempel homomorphism

We next show how, using the inverse of the Lempel homomorphism, a subgraph of $B_k(n-1)$ with certain special properties can be used to construct an antisymmetric Eulerian subgraph of $B_k(n)$.

## 5.1 Preliminaries

We first define the Lempel homomorphism $D$. We give a simplified version of the definition of Alhakim and Akinwande [1], who generalised the original Lempel definition [10].

**Definition 5.1.** Let $D : \mathbb{Z}_k^n \to \mathbb{Z}_k^{n-1}$ be defined such that

$$D(\mathbf{a}) = (a_1 - a_0, a_2 - a_1, \ldots, a_{n-1} - a_{n-2})$$

where $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1}) \in \mathbb{Z}_k^n$. Clearly $D$ is onto; thus we also define $D^{-1}$ to map an element of $\mathbb{Z}_k^{n-1}$ to the set of its pre-images under $D$.

We have the following.

**Lemma 5.1** (Alhakim & Akinwande, [1])**.** $|D^{-1}(\mathbf{a})| = k$ *for every* $\mathbf{a} \in \mathbb{Z}_k^{n-1}$.

Since the edges of $B_k(n-1)$ are associated with $k$-ary $n$-tuples, $D$ induces a mapping, which we also write as $D$, from $B_k(n)$ to $B_k(n-1)$. It is simple to show that $D$ is a graph homomorphism. In this context we also write $D^{-1}$ for the pre-image mapping which maps an edge of $B_k(n-1)$ to a set of $k$ edges of $B_k(n)$ and each pre-image edge contains corresponding pre-image vertices.

## 5.2 Using the inverse homomorphism

The following definition is key to the construction.

**Definition 5.2.** Suppose $T$ is a subgraph of the de Bruijn digraph $B_k(n-1)$ for some $n \geq 2$ and $k \geq 2$. $T$ is said to be *antinegasymmetric* if the following property holds.

Suppose $\mathbf{x} = (x_0, x_1, \ldots, x_{n-2}), \mathbf{y} = (y_0, y_1, \ldots, y_{n-2}) \in \mathbb{Z}_k^{n-1}$ are vertices in $T$. Then if $(\mathbf{x}, \mathbf{y})$ is an edge in $T$, then $(-\mathbf{y}^R, -\mathbf{x}^R)$ is not an edge in $T$.

**Lemma 5.2.** *Suppose $n \geq 2$ and $k \geq 3$. If $T$ is an antinegasymmetric subgraph of the de Bruijn digraph $B_k(n-1)$ with edge set $E$, then $D^{-1}(E)$, of cardinality $k|E|$, is the set of edges for an antisymmetric subgraph $D^{-1}(T)$ of $B_k(n)$. Moreover, the in-degree (out-degree) of any vertex $\mathbf{x}$ in $D^{-1}(T)$ equals the in-degree of $D(\mathbf{x})$ in $T$; thus, in particular, if every vertex of $T$ has in-degree equal to its out-degree, then the same applies to $D^{-1}(T)$.*

*Proof.* Suppose $D^{-1}(T)$ is not antisymmetric, i.e. there exist $(n+1)$-tuples $\mathbf{a} = (a_0, a_1, \ldots, a_n), \mathbf{b} = (b_0, b_1, \ldots, b_n) \in D^{-1}(E)$ such that $\mathbf{a} = \mathbf{b}^R$, i.e. $a_i = b_{n-i}$, for $0 \leq i \leq n$.

Suppose also that $D(\mathbf{a}) = \mathbf{c}$ and $D(\mathbf{b}) = \mathbf{d}$, where $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1}), \mathbf{d} = (d_0, d_1, \ldots, d_{n-1}) \in E$. Hence $c_i = a_{i+1} - a_i$ and $d_i = b_{i+1} - b_i$ for $0 \leq i \leq n-1$. Since $a_i = b_{n-i}$ for $0 \leq i \leq n$, for any $j$ $(0 \leq j \leq n-1)$ we have:

$$c_j = a_{j+1} - a_j = b_{n-(j+1)} - b_{n-j} = -(b_{n-j} - b_{(n-1)-j}) = -d_{(n-1)-j}.$$

Hence $\mathbf{c} = -\mathbf{d}^R$, but this contradicts the assumption that $T$ is antinegasymmetric.

Every edge in $E$ corresponds to $k$ edges in $D^{-1}(E)$, and hence $|D^{-1}(E)| = k|E|$.

It remains to show that every vertex of $D^{-1}(T)$ has in-degree equal to its out-degree. Suppose $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is a vertex of $D^{-1}(T)$. For every edge $\mathbf{a} \in D^{-1}(T)$ that ends in $\mathbf{x}$, there is a corresponding edge $D(\mathbf{a}) \in T$ that ends in $D(\mathbf{x})$. That is, the in-degree of $\mathbf{x}$ in $D^{-1}(T)$ will equal the in-degree of $D(\mathbf{x})$ in $T$. An exactly similar result holds for out-degree. Since every vertex of $T$ has in-degree equal to its out-degree, the same holds for $D^{-1}(T)$. $\qquad\square$

**Example 5.1.** Observe that $E = \{(1,0), (1,1), (1,2), (0,1), (2,1)\}$ is the set of edges for an antinegasymmetric subgraph of the de Bruijn digraph $B_4(1)$. Observe that the edges of $T$ are all the 4-ary 2-tuples containing a 1 and not containing a $3 = -1$; this clearly guarantees antinegasymmetry. It is also simple to see that the in-degree of every vertex is the same as its out-degree.

Then $D^{-1}(E)$ is equal to

$$\{(a, a+1, a+1), (a, a+1, a+2), (a, a+1, a+3), (a, a, a+1), (a, a+2, a+3) : a \in \mathbb{Z}_4\}$$

which is a set of 20 edges, which by Lemma 5.2 forms an antisymmetric subgraph of the de Bruijn digraph $B_4(2)$ with in-degree equal to out-degree for every vertex. It is also simple to check that the graph is connected (ignoring vertices with in-degree zero), and hence is Eulerian. As a result Eulerian circuits exist, each of which corresponds to an $\mathcal{OS}_4(3)$ of period 20. One such orientable sequence is the $\mathcal{OS}_4(3)$ [00112012230130231233] of period 20 found by Gabrić and Sawada [8] via an exhaustive search.

## 5.3 Constructing antinegasymmetric subgraphs

We next demonstrate how to construct antinegasymmetric subgraphs of the de Bruijn digraph $B_k(n - 1)$ for every $n \geq 2$. This builds on the work described in [13].

**Definition 5.3** ([13])**.** Suppose $\mathbf{u} = (u_0, u_1, \ldots, u_{n-1})$ is an $n$-tuple of elements of $\mathbb{Z}_k$ ($k > 1$). Define the function $f : \mathbb{Z}_k \to \mathbb{Q}$ as follows: for any $u \in \mathbb{Z}_k$ treat $u$ as an integer in the range $[0, k-1]$ and set $f(u) = u$ if $u \neq 0$ and $f(u) = k/2$ if $u = 0$. Then the *pseudoweight* of $\mathbf{u}$ is defined to be the sum

$$w^*(\mathbf{u}) = \sum_{i=0}^{n-1} f(u_i)$$

where the sum is computed in $\mathbb{Q}$.

As a simple example for $k = 3$, the 4-tuple $(0, 1, 1, 2)$ has pseudoweight $1.5 + 1 + 1 + 2 = 5.5$, since $f(0) = \frac{3}{2}$.

**Definition 5.4.** Let $E_k(n)$ be the set of all $k$-ary $n$-tuples with pseudoweight less than $nk/2$.

Following [13] we also make the following definition.

**Definition 5.5.** If $k \geq 2$ and $n \geq 1$, let $r_{k,n,s}$ denote the number of $k$-ary $n$-tuples with pseudoweight exactly $s$, where $r_{k,n,s} = 0$ by definition if $s < n$ or $s > n(k-1)$.

A discussion of the properties of $r_{k,n,s}$ and a table of small values is given in [13, Section 3.4]. We can now give the following key result that is closely related to [13, Theorem 3.14].

**Theorem 5.3.** *Suppose $n \geq 2$ and $k \geq 3$. $E_k(n)$ is the set of edges for an antinegasymmetric subgraph of the de Bruijn digraph $B_k(n-1)$ containing $\frac{k^n - r_{k,n,nk/2}}{2}$ edges. Moreover, every vertex in this subgraph has in-degree equal to its out-degree.*

*Proof.* Consider any $n$-tuple $\mathbf{u} = (u_0, u_1, \ldots, u_{n-1}) \in E_k(n)$. By definition we know that $w^*(\mathbf{u}) < nq/2$. We claim that $w^*(-\mathbf{u}^R) = nk - w^*(\mathbf{u})$. This follows immediately from the definition of $f$ since $f(-u_i) = k - f(u_i)$ for every possible value of $u_i$.

Hence, since $w^*(\mathbf{u}) < nk/2$, it follows immediately that $w^*(-\mathbf{u}^R) > nk/2$. Thus the $n$-tuples in $E_k(n)$ are all distinct from the $n$-tuples in $-E_k(n)^R = \{-\mathbf{u}^R : \mathbf{u} \in E_k(n)\}$. Hence $E_k(n)$ is the set of edges for an antinegasymmetric subgraph of the de Bruijn digraph $B_k(n-1)$. The cardinality of $E_k(n)$ is $\frac{k^n - r_{k,n,nk/2}}{2}$ since there is a trivial 1-1 correspondence between the $n$-tuples in $E_k(n)$ and those with pseudoweight greater than $kn/2$.

It remains to show that the in-degree of every vertex is equal to its out-degree. Consider any vertex $\mathbf{u} = (u_0, u_1, \ldots, u_{n-2})$ of $B_k(n-1)$. An incoming edge

$$(s, u_0, u_1, \ldots, u_{n-2})$$

in $E_k(n)$ must satisfy $s + w^*(\mathbf{u}) < nk/2$. Similarly an outgoing edge

$$(u_0, u_1, \ldots, u_{n-2}, t)$$

in $E_k(n)$ must satisfy $t + w^*(\mathbf{u}) < nk/2$. That is, the in-degree of every vertex is the same as its out-degree. $\square$

From Lemma 5.2, this means that $D^{-1}(E_k(n))$ is the set of edges for an antisymmetric subgraph of $B_k(n)$, where every vertex has in-degree equal to its out-degree. It would be ideal if we could also show that $D^{-1}(E_k(n))$ is the set of edges for a *connected* subgraph of $B_k(n)$ and then we would know that $D^{-1}(E_k(n))$ is the set of edges for an Eulerian antisymmetric subgraph

of $B_k(n)$. We could then apply Theorem 3.2 to obtain an $\mathcal{OS}_k(n+1)$ of period $|D^{-1}(E_k(n))| = k|E_k(n)|$.

Thus, showing that the subgraph is connected, after removal of any vertices with in-degree and out-degree zero, is key. This is our next focus.

## 5.4 Establishing connectivity

We next establish the key result that $D^{-1}(E_k(n))$ is the set of edges for a *connected* subgraph of $B_k(n)$. We first need the following simple lemma.

**Lemma 5.4.** *Suppose $n \geq 2$ and $k \geq 3$. If $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ is a vertex in the subgraph of $B_k(n)$ for which $D^{-1}(E_k(n))$ is the set of edges, then $\mathbf{a}$ has in-degree and out-degree zero if and only if*

$$w^*(D(\mathbf{a})) = w^*((a_1 - a_0, a_2 - a_1, \ldots, a_{n-1} - a_{n-2})) \geq \frac{nk}{2} - 1.$$

*Proof.* Suppose $(x, a_0, a_1, \ldots, a_{n-1})$ is an incoming edge to the vertex $\mathbf{a}$, for some $x$. Then $D((x, a_0, a_1, \ldots, a_{n-1}))$ must be an edge in the subgraph of $B_k(n-1)$ for which $E_k(n)$ is the set of edges. Now $D((x, a_0, a_1, \ldots, a_{n-1})) = (a_0 - x, a_1 - a_0, a_2 - a_1, \ldots, a_{n-1} - a_{n-2})$ which must be an edge in $E_k(n)$. Thus, by definition of $E_k(n)$:

$$w^*((a_0 - x, a_1 - a_0, a_2 - a_1, \ldots, a_{n-1} - a_{n-2})) < nk/2$$

It is straightforward to see that

$$w^*((a_0 - x, a_1 - a_0, a_2 - a_1, \ldots, a_{n-1} - a_{n-2})) =$$
$$w^*((a_0 - x)) + w^*((a_1 - a_0, a_2 - a_1, \ldots, a_{n-1} - a_{n-2})).$$

and $w^*((a_0 - x)) \geq 1$; hence

$$w^*((a_1 - a_0, a_2 - a_1, \ldots, a_{n-1} - a_{n-2})) < nk/2 - 1.$$

That is, $\mathbf{a}$ has in-degree and out-degree zero if

$$w^*((a_1 - a_0, a_2 - a_1, \ldots, a_{n-1} - a_{n-2})) \geq nk/2 - 1.$$

To show that $\mathbf{a}$ has non-zero in-degree and out-degree if

$$w^*((a_1 - a_0, a_2 - a_1, \ldots, a_{n-1} - a_{n-2})) < nk/2 - 1$$

it suffices to point out that the edge $D((a_0 - 1, a_0, a_1, \ldots, a_{n-1}))$ will have pseudoweight less than $1 + (nk/2 - 1) = nk/2$, and hence $(a_0 - 1, a_0, a_1, \ldots, a_{n-1})$ is an edge in $E_k(n)$. The result follows. $\square$

We can now establish the main result.

**Theorem 5.5.** *Suppose $n \geq 2$ and $k \geq 3$. $D^{-1}(E_k(n))$ is the set of edges of a connected subgraph of $B_k(n)$.*

*Proof.* By definition, the edges of $E_k(n)$ consist of $k$-ary $n$-tuples with pseudoweight less than $kn/2$. Suppose $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ and $\mathbf{b} = (b_0, b_1, \ldots, b_{n-1})$ are vertices in the subgraph of $B_k(n)$ for which $D^{-1}(E_k(n))$ is the set of edges, satisfying $w^*(D(\mathbf{a})) < nk/2 - 1$ and $w^*(D(\mathbf{b})) < nk/2 - 1$. We need to show that there is a path from $\mathbf{a}$ to $\mathbf{b}$ in this subgraph.

The proof relies on three simple observations.

- **Observation A**. Suppose $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1})$ is any vertex in the subgraph with non-zero in-degree, i.e. $w^*(D(\mathbf{c})) < nk/2 - 1$. Consider the $(n+1)$-tuple $\mathbf{c}^+ = (c_0, c_1, \ldots, c_{n-1}, c_{n-1} + 1)$. Now, by definition, $\mathbf{c}^+$ is an element of $D^{-1}(E_k(n))$ if and only if $w^*(D(\mathbf{c}^+)) < nk/2$. However, trivially,

$$w^*(D(\mathbf{c}^+)) = w^*(D(\mathbf{c})) + 1 < (nk/2 - 1) + 1 = nk/2.$$

  Hence $\mathbf{c}^+$ is always an element of $D^{-1}(E_k(n))$, and so there is always an edge from $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1})$ to $(c_1, c_2, \ldots, c_{n-1}, c_{n-1} + 1)$, as long as $\mathbf{c}$ has non-zero out-degree.

- **Observation B**. Suppose $d$ is an arbitrary element of $Z_k$, and consider the $(n+1)$-tuple $\mathbf{d} = (d, d+1, \ldots, d+n-1, d+n)$. It is simple to verify that

$$w^*(D(\mathbf{d})) = n.$$

  Now, since $k \geq 3$, $w^*(D(\mathbf{d})) < nk/2$, and hence $\mathbf{d}$ is an element of $D^{-1}(E_k(n))$. Thus there is always an edge from $(d, d+1, \ldots, d+n-1)$ to $(d+1, d+2, \ldots, d+n)$ for any $d$, and hence there is always a directed path from $(d, d+1, \ldots, d+n-1)$ to $(e, e+1, \ldots, e+n-1)$ for any $d$ and $e$.

- **Observation C**. Suppose $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1})$ is any vertex in the subgraph with non-zero in-degree, i.e. $w^*(D(\mathbf{c})) < nk/2 - 1$. Consider the $(n+1)$-tuple $\mathbf{c}^- = (c_0 - 1, c_0, c_1, \ldots, c_{n-1})$. As previously

$$w^*(D(\mathbf{c}^-)) = w^*(D(\mathbf{c})) + 1 < (nk/2 - 1) + 1 = nk/2.$$

  Hence there is always an edge from $(c_0 - 1, c_0, c_1, \ldots, c_{n-2})$ to $(\mathbf{c} = (c_0, c_1, \ldots, c_{n-1})$, as long as $\mathbf{c}$ has non-zero in-degree.

The proof now follows in three stages.

- Applying Observation A $n-1$ times, there exists a directed path from $\mathbf{a}$ to the vertex $(a_{n-1}, a_{n-1} + 1, \ldots, a_{n-1} + (n-1))$.

- From Observation B, there exists a directed path from $(a_{n-1}, a_{n-1} + 1, \ldots, a_{n-1} + n - 1)$ to $(b_0 - n + 1, b_0 - n + 2, \ldots, b_0)$.

- Applying Observation C $n - 1$ times, there exists a directed path from $(b_0 - n + 1, b_0 - n + 2, \ldots, b_0)$ to $\mathbf{b} = (b_0, b_1, \ldots, b_{n-1})$.

That is, there exists a directed path from $\mathbf{a}$ to $\mathbf{b}$, and the result follows. $\square$

Combining Theorem 5.3 with Theorems 3.2 and 5.5 gives the following.

**Corollary 5.6.** *If $k \geq 3$ and $n \geq 2$ there exists an $\mathcal{OS}_k(n+1)$ with period*

$$k \frac{k^n - r_{k,n,nk/2}}{2}.$$

*Remark* 5.1. We have defined a set of edges forming an Eulerian subgraph of $B_k(n)$, and every Eulerian circuit in this subgraph will yield an $\mathcal{OS}_k(n+1)$. This approach will thus yield many different such sequences, since there will be many possible Eulerian circuits.

**Example 5.2.** As an example of Corollary 5.6, consider the case $k = 3$ and $n = 3$. The ten 3-ary 3-tuples having pseudoweight less than 4.5 are listed in Table 3 — these form the set $E$. The set $D^{-1}(E_k(n))$ consists of the 30 4-tuples given in Table 4, where the 4-tuples are grouped in threes according to the element of $E_k(n)$ of which they are pre-images under $D$. Finally, a period of an $\mathcal{OS}_5(3)$ containing the 4-tuples in $D^{-1}(E_k(n))$ is:

$$[01201\ 21202\ 01012\ 22011\ 20011\ 12200]$$

.

Table 3: $E_k(n)$: 3-ary 3-tuples with pseudoweight less than 4.5

| | | |
|---|---|---|
| 111 | | |
| 011 | 101 | 110 |
| 001 | 010 | 100 |
| 112 | 121 | 211 |

Table 4: $D^{-1}(E_k(n))$: 3-ary 4-tuples

| 0120 | 1201 | 2012 | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0012 | 1120 | 2201 | 0112 | 1220 | 2001 | 0122 | 1200 | 2011 |
| 0001 | 1112 | 2220 | 0011 | 1122 | 2200 | 0111 | 1222 | 2000 |
| 0121 | 1202 | 2010 | 0101 | 1212 | 2020 | 0201 | 1012 | 2120 |

The periods of the sequences of both Corollary 5.6 (given first) and Corollary 4.3 (given second) for small $k$ and $n$ are given in Table 5, along with the bound on the period from Theorem 2.13 in brackets. In the cases $n = 3$ and $n = 4$ ($k$ odd), the periods of the sequences of Corollary 5.6 meet the bound of Theorem 2.13.

Table 5: Periods of the constructed $\mathcal{OS}_k(n)$ (and bounds)

| $n$ | $k = 3$ | $k = 4$ | $k = 5$ | $k = 6$ | $k = 7$ | $k = 8$ |
|---|---|---|---|---|---|---|
| 2 | – | – | – | – | – | – |
| | – | – | 10 | 12 | 21 | 24 |
| | (3) | (4) | (10) | (12) | (21) | (24) |
| 3 | 9 | 20 | 50 | 84 | 147 | 216 |
| | – | – | 50 | 72 | 147 | 192 |
| | (9) | (20) | (50) | (84) | (147) | (216) |
| 4 | 30 | 88 | 280 | 534 | 1134 | 1800 |
| | – | – | 250 | 432 | 1029 | 1536 |
| | (30) | (112) | (280) | (612) | (1134) | (1984) |
| 5 | 93 | 372 | 1390 | 3300 | 7763 | 14680 |
| | – | – | 1250 | 2592 | 7203 | 12288 |
| | (99) | (452) | (1450) | (3684) | (8085) | (15896) |
| 6 | 288 | 1544 | 7160 | 20172 | 56056 | 118864 |
| | – | – | 6250 | 15552 | 50421 | 98304 |
| | (315) | (1958) | (7550) | (23019) | (58065) | (130332) |
| 7 | 882 | 6344 | 35810 | 122646 | 388626 | 959160 |
| | – | – | 31250 | 93312 | 352947 | 786432 |
| | (972) | (7844) | (38100) | (138144) | (408072) | (1042712) |
| 8 | 2691 | 25904 | 181100 | 743370 | 2757937 | 7724552 |
| | – | – | 156250 | 559872 | 2470629 | 6291456 |
| | (3096) | (32390) | (193800) | (837879) | (2876496) | (8382492) |

# 6  Relation to other work

It is of interest to consider how the construction methods described here affect efforts to find orientable sequences with the largest possible period. The current state of knowledge for small $n$ and $k > 2$ in this direction is summarised in Table 6, where the upper bound from Theorem 2.13 is given in brackets beneath the largest known period.

The following observations can be made about this table. The bound values (in brackets) follow from Theorem 2.13. The values in bold represent maximal values.

- $n = 2$: The fact that there exists an $\mathcal{OS}_k(2)$ with period meeting

Table 6: Largest known periods for an $\mathcal{OS}_k(n)$ (and bounds)

| $n$ | $k=3$ | $k=4$ | $k=5$ | $k=6$ | $k=7$ | $k=8$ |
|---|---|---|---|---|---|---|
| 2 | **3** | **4** | **10** | **12** | **21** | **24** |
|   | (3) | (4) | (10) | (12) | (21) | (24) |
| 3 | **9** | **20** | **50** | **84** | **147** | **216** |
|   | (9) | (20) | (50) | (84) | (147) | (216) |
| 4 | **30** | 88 | **280** | 534 | **1134** | 1800 |
|   | (30) | (112) | (280) | (612) | (1134) | (1984) |
| 5 | 93 | 372 | 1390 | 3360 | 7763 | 15120 |
|   | (99) | (452) | (1450) | (3684) | (8085) | (15896) |
| 6 | 288 | 1608 | 7160 | 21150 | 56056 | 124320 |
|   | (315) | (1958) | (7550) | (23019) | (58065) | (130332) |
| 7 | 882 | 7308 | 36890 | 135450 | 403389 | 1034264 |
|   | (972) | (7844) | (38100) | (138144) | (408072) | (1042712) |
| 8 | 2691 | 30300 | 187980 | 821940 | 2844408 | 8315496 |
|   | (3096) | (32390) | (193800) | (837879) | (2876496) | (8382492) |

the bound follows from [2, Theorem 5.4] (for $k$ prime), and from [8, Theorem 2] and [13, Lemma 2.2] for general $k$.

- $n = 3$: The existence of an $\mathcal{OS}_k(3)$ meeting the period bound is due to [2, Example 5.1] for $k = 3$, [8, Section 1] for $k = 5$ (from an exhaustive search), and for general $k$ from this paper.

- $n = 4$: The fact that the maximum period of an $\mathcal{OS}_3(4)$ is 30 is again due to [8, Section 1], and the existence of an $\mathcal{OS}_k(4)$ meeting the bound for general odd $k$ is from this paper. All values for $4 \leq k \leq 8$ follow from this paper.

- $n = 5$: The $\mathcal{OS}_3(5)$, $\mathcal{OS}_4(5)$, $\mathcal{OS}_5(5)$ and $\mathcal{OS}_7(5)$, respectively of periods 93, 372, 1390 and 7763, come from this paper. The values for $k = 6$ and $k = 8$ come from [8, Theorem 11] from cycle-joining.

- $n \geq 6$: The $\mathcal{OS}_3(6)$, $\mathcal{OS}_5(6)$, $\mathcal{OS}_3(7)$ and $\mathcal{OS}_3(8)$, of periods 288, 7160, 882 and 2691 respectively, come from this paper. All other values in the table come from [8, Theorem 11].

# 7 Conclusions

We have presented tighter upper bounds on the period of an orientable sequence and a general approach for constructing orientable sequences for any $n \geq 2$ and any $k \geq 3$. For $n \leq 3$, and $n = 4$ when $k$ is odd, the results show that the bound on the period is tight. The construction method gives

sequences of slightly lesser period than the approach of Gabrić and Sawada [8, Theorem 11] for larger $n$, although all the sequences constructed here have period greater than $(k-1)/k$ of the maximum for odd $k$, and greater than $(k-2)/k$ of the maximum for even $k$.

Table 6 reveals that further work is required to close the gap between the known largest period and the best existing upper bound for $k$ even and $n = 4$ as well as for all $k$ when $n > 4$.

# References

[1] A. Alhakim and M. Akinwande, *A recursive construction of nonbinary de Bruijn sequences*, Des. Codes Cryptogr. **60** (2011), no. 2, 155–169.

[2] A. Alhakim, C. J. Mitchell, J. Szmidt, and P. R. Wild, *Orientable sequences over non-binary alphabets*, Cryptogr. Commun. **16** (2024), 1309–1326.

[3] J. Burns and C. J. Mitchell, *Coding schemes for two-dimensional position sensing*, Tech. Report HPL–92–19, January 1992, `https://www.chrismitchell.net/HPL-92-19.pdf`.

[4] ———, *Coding schemes for two-dimensional position sensing*, Cryptography and Coding III (M. J. Ganley, ed.), Oxford University Press, 1993, pp. 31–66.

[5] Z.-D. Dai, K. M. Martin, M. J. B. Robshaw, and P. R. Wild, *Orientable sequences*, Cryptography and Coding III (M. J. Ganley, ed.), Oxford University Press, Oxford, 1993, pp. 97–115.

[6] D. Gabrić and J. Sawada, *Construction of orientable sequences in $O(1)$-amortized time per bit*, 2024, Available at `https://arxiv.org/abs/2401.14341`.

[7] ———, *Efficient construction of long orientable sequences*, 35th Annual Symposium on Combinatorial Pattern Matching, CPM 2024, June 25–27, 2024, Fukuoka, Japan (S. Inenaga and S. J. Puglisi, eds.), LIPIcs, vol. 296, Schloss Dagstuhl — Leibniz-Zentrum für Informatik, 2024, pp. 15:1–15:12.

[8] ———, *Constructing k-ary orientable sequences with asymptotically optimal length*, Des. Codes Cryptogr. **93** (2025), 2349–2367.

[9] A. Gibbons, *Algorithmic graph theory*, Cambridge University Press, Cambridge, 1985.

[10] A. Lempel, *On a homomorphism of the de Bruijn graph and its application to the design of feedback shift registers*, IEEE Transactions on Computers **C-19** (1970), 1204–1209.

[11] C. J. Mitchell and P. R. Wild, *Constructing orientable sequences*, IEEE Trans. Inform. Theory **68** (2022), 4782–4789.

[12] _____, *Special orientable sequences*, 2024, Available at `https://arxiv.org/abs/2411.17273`.

[13] _____, *Orientable and negative orientable sequences*, Discrete Appl. Math. **377** (2025), 242–259.

[14] G. Rauzy, *Suites à termes dans un alphabet fini*, Seminaire de Théorie des Nombres de Bordeaux **12** (1982–1983), 1–16.