New upper bounds for the period of a negative orientable sequence

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Abstract. Negative orientable sequences, i.e. periodic sequences with elements from a finite alphabet of size at least three in which an n-tuple or the negative of its reverse appears at most once in a period of the sequence, were introduced by Alhakim et al. in 2024. The main goal in defining them was as a means of generating orientable sequences, which have automatic position location applications, although they are potentially of interest in their own right. In this paper we develop new upper bounds on the period of negative orientable sequences which, for $n > 2$, are significantly sharper than the previous known bound. The approach used to develop the new bounds involves examining the nodes in the subgraph of the de Bruijn graph corresponding to a negative orientable sequence, and to consider the implications of the fact that the in-degree of every vertex in this subgraph must equal the out-degree. However, despite improving the bounds, a gap remains between the largest known period for a negative orientable sequence and the corresponding bounds for every $n > 2$.

Keywords: orientable sequence · universal sequence · position location · de Bruijn graph

1 Introduction

Orientable sequences are periodic sequences with elements from an alphabet of size k with the property that, for some n (the *order*), any n-tuple occurs at most once in a period of the sequence in either direction. Clearly an orientable sequence of order n, an $\mathcal{OS}_k(n)$, is also an $\mathcal{OS}_k(m)$ for every $m \geq n$. Orientable sequences were introduced in the early 1990s [2,3] in connection with position location applications.

The binary case was studied by Dai et al. [4], who presented an upper bound and a construction method yielding sequences with asymptotically optimal period. Further methods of construction for this case appeared much more recently [5,6,8]. The general alphabet case (i.e. for $k \geq 2$) was first considered by Alhakim et al. [1]. Subsequently, construction methods for this case have been described, [7,9,11], and upper bounds for the period of a k-ary orientable sequence were given in [9] and improved in [11]. Although the construction method of Gabrić and Sawada [7] has been shown to yield sequences of asymptotically optimal period, the problem of determining the precise value of the largest period for an $\mathcal{OS}_k(n)$ remains an open question except for small values of n — for further details of the current state of the art see [11].

Alhakim at al. [1] introduced the notion of a *negative orientable sequence*, the main focus of this paper, to help construct new orientable sequences. A negative orientable sequence of order n, $\mathcal{NOS}_k(n)$, is a periodic k-ary sequence in which an n-tuple or the negative of its reverse appears at most once in a period of the sequence. Such sequences were subsequently studied further in [9,11], where a range of methods of construction are described, and an upper bound on the period of a $\mathcal{NOS}_k(n)$ is given in [11]. Much like the orientable sequence case, determining the precise value of the largest period for an $\mathcal{NOS}_k(n)$ remains unresolved, in this case even for small n.

In this paper we use the same approach as employed in [11] to develop new period upper bounds for orientable sequences to develop new and sharper bounds for the period of a negative orientable sequence. The main tool is to examine nodes in the subgraph of the de Bruijn graph corresponding to a negative orientable sequence, and to consider the implications of the fact that the in-degree of every vertex in this subgraph must equal the out-degree.

The remainder of the paper is structured as follows. Section 2 gives fundamental definitions, some elementary counting results for n-tuples, and a brief introduction to the de Bruijn graph. This is followed in Section 3 by the development of the new period bounds. Finally, Section 4 concludes the paper, summarising the state of knowledge regarding the maximum period for negative orientable sequences.

2 Preliminaries

2.1 Basic Definitions

We need the following key definitions, largely following the notational conventions of [9]. We refer throughout to a k-ary n-tuple to mean a sequence of length n of symbols drawn from $\mathbb{Z}_k$. Note that we assume that $k > 2$ throughout since the negative of a tuple equals the tuple when $k = 2$.

Since we are interested in tuples occurring either forwards or backwards in a sequence, we introduce the notion of a reversed tuple, so that if $\mathbf{u} = (u_0, u_1, \ldots, u_{n-1})$ is a k-ary n-tuple then $\mathbf{u}^R = (u_{n-1}, u_{n-2}, \ldots, u_0)$ is its *reverse*. We are also interested in negating all the elements of a tuple, and hence if $\mathbf{u} = (u_0, u_1, \ldots, u_{n-1})$ is a k-ary n-tuple, we write $-\mathbf{u}$ for $(-u_0, -u_1, \ldots, -u_{n-1})$. We write $\mathbf{s}_n(i)$ for the n-tuple occurring at position i in a sequence $S = (s_i)$.

Definition 1 ([1]). *A k-ary* n-window sequence $S = (s_i)$ *is a periodic sequence of elements from $\mathbb{Z}_k$ ($k > 1$, $n > 1$) with the property that no n-tuple appears more than once in a period of the sequence, i.e. with the property that if $\mathbf{s}_n(i) = \mathbf{s}_n(j)$ for some i, j, then $i \equiv j \pmod{m}$ where m is the period of the sequence.*

This paper is concerned with a special class of n-window sequences: negative orientable sequences.

Definition 2 ([1]). *A k-ary n-window sequence $S = (s_i)$ is said to be a* negative orientable sequence of order n *(a $\mathcal{NOS}_k(n)$) if $\mathbf{s}_n(i) \neq -\mathbf{s}_n(j)^R$, for any i, j.*

Definition 3 ([9]). *Suppose $n \geq 1$ and $k > 2$. A k-ary n-tuple $\mathbf{u} = (u_0, u_1, \ldots, u_{n-1})$ is said to be* negasymmetric *if $u_i = -u_{n-1-i}$ for every i, $0 \leq i \leq n-1$, i.e. if $\mathbf{u} = -\mathbf{u}^R$.*

Clearly an $\mathcal{NOS}_k(n)$ cannot contain any negasymmetric n-tuples. This observation yields a very simple bound, but [9] gives a sharper bound, as follows.

Theorem 1 (Theorem 3.5 of [9]). *Suppose that $S = (s_i)$ is a $\mathcal{NOS}_k(n)$ ($k \geq 2$, $n \geq 2$). Then the period of S is at most*

$$\begin{aligned} (k^n - k^{\lfloor n/2 \rfloor} - k^{\lfloor (n-1)/2 \rfloor} + 1)/2 & \quad \text{if } k \text{ is odd,} \\ (k^n - 2k^{(n-1)/2})/2 - 1 & \quad \text{if } k \text{ is even and } n \text{ is odd,} \\ (k^n - k^{n/2})/2 - 1 & \quad \text{if } k \text{ is even and } n \text{ is even.} \end{aligned}$$

The main goal of this paper is to give a significantly sharper bound. A comparison of the new and old bounds is provided in Table 1.

Definition 4. *Suppose $n \geq 2$ and $k \geq 2$. If $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ is a k-ary n-tuple, then $\mathbf{a}$ is said to be* uniform *if and only if $a_0 = a_1 = \cdots = a_{n-1}$.*

Definition 5. *Suppose $n \geq 2$ and $k \geq 2$. A k-ary n-tuple $\mathbf{u} = (u_0, u_1, \ldots, u_{n-1})$ is said to be* alternating *if and only if there exist c_0 and c_1 ($c_0 \neq c_1$) such that $u_{2i} = c_0$ and $u_{2i+1} = c_1$ for every i.*

Remark 1. Since the above definition requires $c_0 \neq c_1$, an alternating n-tuple is always non-uniform.

Definition 6. *Suppose $n \geq 2$ and $k > 2$. A k-ary n-tuple $\mathbf{u} = (u_0, u_1, \ldots, u_{n-1})$ is said to be* negatively-alternating *if and only if $u_{i+1} = -u_i$ for every i, $0 \leq i \leq n-2$.*

Remark 2. A negatively-alternating n-tuple is uniform if $u_0 = 0$ or $u_0 = k/2$ (k even).

Definition 7. *Suppose $n \geq 2$ and $k > 2$. An n-tuple $(a_0, a_1, \ldots, a_{n-1})$ is said to be* left-semi-negasymmetric *(or* left-sns *for short) if $a_i = -a_{n-i-2}$, $0 \leq i \leq n-2$. Equivalently, $(a_0, a_1, \ldots, a_{n-1})$ is left-sns if and only if $(a_0, a_1, \ldots, a_{n-2})$ is negasymmetric.*

Analogously, an n-tuple $(a_0, a_1, \ldots, a_{n-1})$ is said to be right-semi-negasymmetric *(or* right-sns *for short) if $a_i = -a_{n-i}$, $1 \leq i \leq n-1$. Equivalently, $(a_0, a_1, \ldots, a_{n-1})$ is right-sns if and only if $(a_1, a_2, \ldots, a_{n-1})$ is negasymmetric.*

2.2 Counting Sets of n-tuples

The following simple results will be of use below.

Lemma 1. *Suppose $n \geq 1$ and $k > 2$. Then:*

i) the number of k-ary negasymmetric n-tuples is

$$\begin{aligned} k^{(n-1)/2} &\quad \text{if } n \text{ is odd and } k \text{ is odd} \\ 2k^{(n-1)/2} &\quad \text{if } n \text{ is odd and } k \text{ is even;} \\ k^{n/2} &\quad \text{if } n \text{ is even;} \end{aligned}$$

ii) the number of uniform k-ary n-tuples is k;
iii) the number of negatively-alternating k-ary n-tuples is k;
iv) the number of n-tuples that are both uniform and negatively-alternating is 1 if k is odd and 2 if k is even;
v) the number of uniform negasymmetric k-ary n-tuples is 1 if k is odd and 2 if k is even;
vi) the number of negatively-alternating negasymmetric k-ary n-tuples is 1 if n and k are odd, 2 if n is odd and k is even, and k if n is even;
vii) the number of alternating negasymmetric k-ary n-tuples is zero if n and k are both odd, 2 if n is odd and k is even, $k-1$ if n is even and k is odd, and $k-2$ if n and k are both even (and in every case they are negatively-alternating);
viii) the number of left-sns k-ary n-tuples is:

$$\begin{aligned} k^{(n+1)/2} &\quad \text{if } n \text{ is odd;} \\ k^{n/2} &\quad \text{if } n \text{ is even and } k \text{ is odd;} \\ 2k^{n/2} &\quad \text{if } n \text{ and } k \text{ are both even;} \end{aligned}$$

ix) the number of non-uniform left-sns k-ary n-tuples is:

$$\begin{aligned} k^{(n+1)/2} - 1 &\quad \text{if } n \text{ and } k \text{ are both odd;} \\ k^{(n+1)/2} - 2 &\quad \text{if } n \text{ is odd and } k \text{ is even;} \\ k^{n/2} - 1 &\quad \text{if } n \text{ is even and } k \text{ is odd;} \\ 2k^{n/2} - 2 &\quad \text{if } n \text{ and } k \text{ are both even;} \end{aligned}$$

x) the number of non-negatively-alternating left-sns k-ary n-tuples is:

$$\begin{aligned} k^{(n+1)/2} - k &\quad \text{if } n \text{ is odd;} \\ k^{n/2} - 1 &\quad \text{if } n \text{ is even and } k \text{ is odd;} \\ 2k^{n/2} - 2 &\quad \text{if } n \text{ and } k \text{ are both even.} \end{aligned}$$

xi) the number of non-uniform non-alternating left-sns k-ary n-tuples is:

$$\begin{aligned} k^{(n+1)/2} - k &\quad \text{if } n \text{ is odd;} \\ k^{n/2} - 1 &\quad \text{if } n \text{ is even and } k \text{ is odd;} \\ 2k^{n/2} - 4 &\quad \text{if } n \text{ and } k \text{ are both even;} \end{aligned}$$

Finally observe that (viii), (ix), (x) and (xi) also hold if left-sns is replaced with right-sns.

Proof. (i) is Lemma 3.1 of [9]. (ii), (iii) and (iv) are immediate. (v) follows by observing that a uniform negasymmetric n-tuple must have every entry equal to zero or $k/2$.

For (vi), if n is odd then every element in the n-tuple must be either 0 or $k/2$ and hence the n-tuple must be uniform; the result follows from (v). If n is even then every negatively-alternating n-tuple is negasymmetric, and the result follows from (iii).

For (vii), if n is odd then every element in the n-tuple must be either 0 or $k/2$, and the result follows from (v) since an alternating n-tuple cannot be uniform. If n is even then an alternating negasymmetric n-tuple must be negatively-alternating, and the result follows from (vi) and (iv) since an alternating n-tuple cannot be uniform.

(viii) follows from (i) and observing that an n-tuple is left-sns if and only if its first $n-1$ entries form a negasymmetric $(n-1)$-tuple. A left-sns n-tuple can only be uniform if it is the all-zero or the all-$k/2$ n-tuple, and (ix) follows from (v) and (viii).

For (x), If n is odd, then a negatively-alternating n-tuple is left-sns and in this case the number is simply (viii) minus (iii). If n is even, a negatively-alternating n-tuple is only left-sns if it is also uniform, i.e. the number is (viii) minus (iv).

Finally, for (xi), first observe that an n-tuple cannot be both alternating and uniform by definition, hence (xi) is simply (ix) less the number of alternating left-sns n-tuples. Also, the number of alternating left-sns n-tuples is the same as the number of alternating negasymmetric $(n-1)$-tuples, i.e., from Lemma 1(vii), zero if n is even and k is odd, 2 if n and k are both even, $k-1$ if n and k are both odd, and $k-2$ if n is odd and k is even. □

2.3 Negative Orientable Sequences and the de Bruijn Digraph

We also need the following definitions. Following Alhakim et al. [1] and many previous authors, we introduce the k-ary de Bruijn digraph.

Definition 8. *For positive integers n and k greater than one, let $\mathbb{Z}_k^n$ be the set of all k^n tuples of length n with entries from the group $\mathbb{Z}_k$ of residues modulo k. The order n* de Bruijn digraph*, $B_k(n)$, is a directed graph with $\mathbb{Z}_k^n$ as its vertex set in which, for any two vertices $\boldsymbol{x} = (x_0, x_1, \ldots, x_{n-1})$ and $\boldsymbol{y} = (y_0, y_1, \ldots, y_{n-1})$, the pair $(\boldsymbol{x}, \boldsymbol{y})$ is an edge if and only if $y_i = x_{i+1}$ for every i $(0 \le i < n-1)$. We label such an edge with the $(n+1)$-tuple $(x_0, x_1, \ldots, x_{n-1}, y_{n-1})$.*

Note that we have defined two ways of specifying an edge in $B_k(n)$, namely as either a pair of vertices $(\mathbf{a}, \mathbf{b})$, where $\mathbf{a}, \mathbf{b}$ are k-ary n-tuples, or as a single k-ary $(n+1)$-tuple $\mathbf{x}$. We refer to an edge in $B_k(n)$ and a k-ary $(n+1)$-tuple interchangeably throughout.

Definition 9. *Suppose $k > 2$ and $n \geq 2$. Let $B_k^-(n-1)$ be the subgraph of the de Bruijn digraph $B_k(n-1)$ with all the edges corresponding to negasymmetric n-tuples removed.*

The following elementary result follows immediately from Lemma 1.

Lemma 2. *If $k > 2$ and $n \geq 2$ the number $N_k(n)$ of edges in $B_k^-(n-1)$ is*

$$\begin{aligned} k^n - k^{(n-1)/2} &\quad \textit{if } n \textit{ is odd and } k \textit{ is odd} \\ k^n - 2k^{(n-1)/2} &\quad \textit{if } n \textit{ is odd and } k \textit{ is even} \\ k^n - k^{n/2} &\quad \textit{if } n \textit{ is even.} \end{aligned}$$

Definition 10. *Suppose $k > 2$ and $n \geq 2$. Suppose S is an $\mathcal{NOS}_k(n)$. Let $B^-(S,n)$ be the subgraph of $B_k(n-1)$ with vertices those of $B_k(n-1)$ and with edges corresponding to those n-tuples which appear in either S or $-S^R$. We refer to $B^-(S,n)$ as the nega-sequence-subgraph.*

The following simple lemma is key.

Lemma 3. *Suppose $k > 2$ and $n \geq 2$. Suppose S is an $\mathcal{NOS}_k(n)$ of period m. Then:*

i) $B^-(S,n)$ contains $2m$ edges;
ii) every vertex of $B^-(S,n)$ has in-degree equal to its out-degree; and
iii) $B^-(S,n)$ is a subgraph of $B_k^-(n-1)$.

Proof. i) Since S has period m, a total of $2m$ n-tuples appear in S and $-S^R$. They are all distinct since S is an $\mathcal{NOS}_k(n)$.
ii) S and $-S^R$ correspond to edge-disjoint Eulerian circuits in $B^-(S,n)$, and the result follows.
iii) This is immediate since an $\mathcal{NOS}_k(n)$ cannot contain any negasymmetric n-tuples.

□

3 The New Bounds

3.1 In-out-degree Constraints on the Nega-sequence-subgraph

In this section and the next we consider properties of the nega-sequence-subgraph for a $\mathcal{NOS}_k(n)$ S. We first have the following.

Lemma 4. *Suppose $k > 2$ and $n \geq 3$. A vertex in $B_k^-(n-1)$ has:*

i) in-degree $k-1$ if and only if its label is left-sns; otherwise it has in-degree k;
ii) out-degree $k-1$ if and only if its label is right-sns; otherwise it has out-degree k.

Proof. For (i), the in-degree of every vertex in $B_k(n-1)$ is k. However, if (and only if) an inbound edge corresponds to a negasymmetric n-tuple, then this edge will not be in $B_k^-(n-1)$. Such an event can occur if and only if the vertex is labelled with a left-sns $(n-1)$-tuple, and there can only be one such negasymmetric inbound edge. The result follows. The proof of (ii) follows using an exactly analogous argument. □

By Lemma 3(ii), this immediately tells us that some edges in $B_k^-(n-1)$ cannot occur in $B^-(S,n)$ if S is a $\mathcal{NOS}_k(n)$. However, before describing exactly when this occurs, we first need the following simple result.

Lemma 5. *Suppose $k>2$ and $n \geq 4$. Suppose the $(n-1)$-tuple $(a_0,a_1,\ldots,a_{n-2})$ is both left-sns and right-sns. Then*

i) if n is odd and k is odd then $(a_0,a_1,\ldots,a_{n-2})$ is the all-zero uniform $(n-1)$-tuple;
ii) if n is odd and k is even then $(a_0,a_1,\ldots,a_{n-2})$ is either uniform or alternating, where $a_i \in \{0,k/2\}$ for every i;
iii) if n is even then $(a_0,a_1,\ldots,a_{n-2})$ is negatively-alternating.

Further, all the $(n-1)$-tuples in (i), (ii) and (iii) are both left-sns and right-sns.

Proof. Suppose the $(n-1)$-tuple $(a_0,a_1,\ldots,a_{n-2})$ is both left-sns and right-sns. Then $a_i=-a_{n-i-3}$, $0 \leq i \leq n-3$, and $a_i=-a_{n-i-1}$, $1 \leq i \leq n-2$. Since $n \geq 4$ this implies that there exist constants c_0 and c_1 such that $c_0=a_{2i}$, $0 \leq 2i \leq n-2$, and $c_1=a_{2j+1}$, $0 \leq 2j+1 \leq n-2$.

If n is even then we have $a_{(n-2)/2}=-a_{(n-2)/2-1}$ (from left-semi-negasymmetry), and hence $c_0=-c_1$, and (iii) follows.

If n is odd then, from left-semi-negasymmetry, $a_{(n-3)/2}=-a_{(n-3)/2}$ and from right-semi-negasymmetry $a_{(n-1)/2}=-a_{(n-1)/2}$, and hence $c_0=-c_0$ and $c_1=-c_1$. If k is odd then we have $c_0=c_1=0$, and (i) follows. if k is even then c_0 and c_1 are both either 0 or $k/2$, and (ii) follows. □

The following result follows immediately from Lemmas 4 and 5.

Corollary 1. *Suppose $k>2$ and $n \geq 4$. Suppose S is an $\mathcal{NOS}_k(n)$ and consider a vertex in $B_k^-(n-1)$ with label $\mathbf{a}=(a_0,a_1,\ldots,a_{n-2})$, where $\mathbf{a}$ is non-uniform.*

i) if n and k are both odd, and $\mathbf{a}$ is left-sns, then its in-degree is $k-1$ and its out-degree is k;
ii) if n and k are both odd and $\mathbf{a}$ is right-sns, then its out-degree is $k-1$ and its in-degree is k;
iii) if n is odd and k is even and $\mathbf{a}$ is left-sns and non-alternating, then its in-degree is $k-1$ and its out-degree is k;
iv) if n is odd and k is even and $\mathbf{a}$ is right-sns and non-alternating, then its in-degree is k and its out-degree is $k-1$;
v) if n is even and $\mathbf{a}$ is left-sns and not negatively-alternating, then its in-degree is $k-1$ and its out-degree is k;

vi) if n is even and $\mathbf{a}$ *is right-sns and not negatively-alternating, then its out-degree is $k-1$ and its in-degree is k.*

The above corollary immediately tells us that certain edges in $B_k^-(n-1)$ cannot occur in $B^-(S,n)$, as follows.

Corollary 2. *Suppose $k > 2$ and $n \geq 4$ and S is an $\mathcal{NOS}_k(n)$. Then*

i) if n and k are both odd, for every vertex in $B_k^-(n-1)$ corresponding to a non-uniform left-sns $(n-1)$-tuple $(a_0, a_1, \ldots, a_{n-2})$, there is an edge $(a_0, a_1, \ldots, a_{n-2}, x)$ in $B_k^-(n-1)$, for some x, that is not in $B^-(S,n)$.
ii) if n and k are both odd, for every vertex in $B_k^-(n-1)$ corresponding to a non-uniform right-sns $(n-1)$-tuple $(a_0, a_1, \ldots, a_{n-2})$, there is an edge $(y, a_0, a_1, \ldots, a_{n-2})$ in $B_k^-(n-1)$, for some y, that is not in $B^-(S,n)$.
iii) if n is odd and k is even, for every vertex in $B_k^-(n-1)$ corresponding to a non-uniform non-alternating left-sns $(n-1)$-tuple $(a_0, a_1, \ldots, a_{n-2})$, there is an edge $(a_0, a_1, \ldots, a_{n-2}, x)$ in $B_k^-(n-1)$, for some x, that is not in $B^-(S,n)$.
iv) if n is odd and k is even, for every vertex in $B_k^-(n-1)$ corresponding to a non-uniform non-alternating right-sns $(n-1)$-tuple $(a_0, a_1, \ldots, a_{n-2})$, there is an edge $(y, a_0, a_1, \ldots, a_{n-2})$ in $B_k^-(n-1)$, for some x, that is not in $B^-(S,n)$.
v) if n is even, for every vertex in $B_k^-(n-1)$ corresponding to a non-negatively-alternating left-sns $(n-1)$-tuple $(a_0, a_1, \ldots, a_{n-2})$, there is an edge $(a_0, a_1, \ldots, a_{n-2}, x)$ in $B_k^-(n-1)$, for some x, that is not in $B^-(S,n)$.
vi) if n is even, for every vertex in $B_k^-(n-1)$ corresponding to a non-negatively-alternating right-sns $(n-1)$-tuple $(a_0, a_1, \ldots, a_{n-2})$, there is an edge $(y, a_0, a_1, \ldots, a_{n-2})$ in $B_k^-(n-1)$, for some y, that is not in $B^-(S,n)$.

Proof. The result follows immediately from Lemma 3(ii) and Corollary 1. □

3.2 Degree-parity Constraints on the Nega-sequence-subgraph

We first give the following simple lemma.

Lemma 6. *Suppose $k > 2$ and $n \geq 2$. If an $(n-1)$-tuple is both left-sns and negasymmetric then it is uniform. Similarly, if an $(n-1)$-tuple is both right-sns and negasymmetric then it is uniform.*

Proof. Let $\mathbf{a} = (a_0, \ldots, a_{n-2})$ be an $(n-1)$-tuple which is both left-sns and negasymmetric. Then $a_i = -a_{n-2-i} = -a_{n-3-i}$ for $i = 0, 1, \ldots, n-2$. It is immediate that $a_j = a_{j+1}$ for $j = 0, 1, \ldots, n-3$ and thus $\mathbf{a}$ is uniform. The second claim follows by an analogous argument. □

The following lemma is key.

Lemma 7. *Suppose $k > 2$ and $n \geq 2$. Suppose S is an $\mathcal{NOS}_k(n)$ and consider a vertex in $B^-(S,n)$ with label* $\mathbf{a} = (a_0, a_1, \ldots, a_{n-2})$, *where* $\mathbf{a}$ *is negasymmetric. Then* $\mathbf{a}$ *has even in-degree and even out-degree in $B^-(S,n)$.*

Proof. Both S and $-S^R$ correspond to an Eulerian circuit in $B^-(S,n)$, and these circuits are edge-disjoint and cover all the edges of $B^-(S,n)$. If $\mathbf{a}$ is negasymmetric then both circuits pass through this vertex equally many times. It follows that $\mathbf{a}$ has even in-degree and even out-degree in $B^-(S,n)$. □

We also need the following.

Lemma 8. *Suppose $k>2$ and $n\geq 2$, and consider a vertex in $B_k^-(n-1)$ with label $\mathbf{a}=(a_0,a_1,\ldots,a_{n-2})$. Then in all the following cases this vertex has odd in-degree and odd out-degree in $B_k^-(n-1)$:*

i) if k is odd and $\mathbf{a}$ is negasymmetric but not uniform;
ii) if n is odd, k is even and $\mathbf{a}$ is either uniform or alternating, where $a_i\in\{0,k/2\}$ for every i;
iii) if n and k are both even and $\mathbf{a}$ is negatively-alternating.

Proof. i) Since $\mathbf{a}$ is negasymmetric but not uniform, by Lemma 6 is cannot be left-sns or right-sns. Thus, by Lemma 4 it has in-degree and out-degree k, and since k is odd the result follows.
ii) Since $\mathbf{a}$ is either uniform or alternating, where $a_i\in\{0,k/2\}$ for every i, by Lemma 5(ii) it is both left-sns and right-sns. Thus, by Lemma 4 it has in-degree and out-degree $k-1$, and since k is even the result follows.
iii) Since $\mathbf{a}$ is negatively-alternating, by Lemma 5(iii) it is both left-sns and right-sns. Thus, by Lemma 4 it has in-degree and out-degree $k-1$. Since k is even the result follows.

□

Combining Lemmas 7 and 8 immediately gives the following important result.

Corollary 3. *Suppose $k>2$ and $n\geq 2$ and S is an $\mathcal{NOS}_k(n)$. Then in all the following cases there are edges $(a_0,a_1,\ldots,a_{n-2},x)$ and $(y,a_0,a_1,\ldots,a_{n-2})$ in $B_k^-(n-1)$ that are not in $B^-(S,n)$.*

i) if k is odd, for every vertex in $B_k^-(n-1)$ corresponding to a negasymmetric non-uniform $(n-1)$-tuple $(a_0,a_1,\ldots,a_{n-2})$;
ii) if n is odd and k is even, for every vertex in $B_k^-(n-1)$ corresponding to a uniform or alternating negasymmetric $(n-1)$-tuple $(a_0,a_1,\ldots,a_{n-2})$, where $a_i\in\{0,k/2\}$ for every i;
iii) if n and k are both even, for every vertex in $B_k^-(n-1)$ corresponding to a negatively-alternating negasymmetric $(n-1)$-tuple $(a,-a,a,-a,\ldots,a)$.

3.3 Constraint Interactions

The results above indicate in what circumstances certain edges in $B_k^-(n-1)$ cannot occur in $B^-(S,n)$ if S is an $\mathcal{NOS}_k(n)$. In particular we have considered cases where one of the incoming edges to a vertex corresponding to a negasymmetric or right-sns $(n-1)$-tuple cannot occur in $B^-(S,n)$, and also where one of the outgoing edges from a vertex corresponding to a negasymmetric or left-sns

$(n-1)$-tuple cannot occur in $B^{-}(S,n)$. While we would like to add together the numbers of eliminated edges for each case, we need to ensure we avoid 'double counting'.

More specifically, we need to consider when an edge can be outgoing from a negasymmetric or left-sns $(n-1)$-tuple and also incoming to a negasymmetric or right-sns $(n-1)$-tuple. This motivates the following result.

Lemma 9. *Suppose $k>2$. Suppose $\mathbf{a}=(a_0,a_1,\ldots,a_{n-2})$ and $\mathbf{b}=(a_1,a_2,\ldots,a_{n-1})$ are k-ary $(n-1)$-tuples.*

i) If $n\geq 3$, and $\mathbf{a}$ and $\mathbf{b}$ are both negasymmetric then
- *if n is even then $\mathbf{a}$ and $\mathbf{b}$ are both uniform or alternating, where $a_i\in\{0,k/2\}$ for every i;*
- *if n is odd then $\mathbf{a}$ and $\mathbf{b}$ are negatively-alternating.*

ii) If $n\geq 5$, $\mathbf{a}$ is negasymmetric and $\mathbf{b}$ is right-sns then there exist c_j, $0\leq j\leq 2$, such that $a_{3i+j}=c_j$, for every i and $0\leq j\leq 2$. Moreover if $n\equiv 0 \pmod 3$ then $c_2\in\{0,k/2\}$ and $c_0=-c_1$; if $n\equiv 1 \pmod 3$ then $c_1\in\{0,k/2\}$ and $c_0=-c_2$; and if $n\equiv 2 \pmod 3$ then $c_0\in\{0,k/2\}$ and $c_1=-c_2$.

iii) If $n\geq 5$, $\mathbf{a}$ is left-sns and $\mathbf{b}$ is negasymmetric then there exist c_j, $0\leq j\leq 2$, such that $a_{3i+j}=c_j$, for every i and $0\leq j\leq 2$. Moreover if $n\equiv 0 \pmod 3$ then $c_0\in\{0,k/2\}$ and $c_1=-c_2$; if $n\equiv 1 \pmod 3$ then $c_2\in\{0,k/2\}$ and $c_0=-c_1$; and if $n\equiv 2 \pmod 3$ then $c_1\in\{0,k/2\}$ and $c_0=-c_2$.

iv) If $n\geq 5$, $\mathbf{a}$ is left-sns and $\mathbf{b}$ is right-sns then there exist c_j, $0\leq j\leq 3$, such that $a_{4i+j}=c_j$, for every i and $0\leq j\leq 3$. Moreover if $n\equiv 0 \pmod 4$ then $c_0=-c_1$ and $c_2=-c_3$; if $n\equiv 1 \pmod 4$ then $c_0=-c_2$, $c_1\in\{0,k/2\}$, and $c_3\in\{0,k/2\}$; if $n\equiv 2 \pmod 4$ then $c_0=-c_3$ and $c_1=-c_2$; and if $n\equiv 3 \pmod 4$ then $c_1=-c_3$, $c_0\in\{0,k/2\}$, and $c_2\in\{0,k/2\}$.

Proof. i) By negasymmetry of $\mathbf{a}$ and $\mathbf{b}$, respectively, we have $a_i=-a_{n-2-i}$ and $a_{i+1}=-a_{n-1-i}$ for every i, $0\leq i\leq n-2$. Hence there exist c_0 and c_1 such that $a_{2i}=c_0$ and $a_{2i+1}=c_1$ for every i.

If n is even then from the first equation we have $a_{n/2-1}=-a_{n/2-1}$, i.e. $a_{n/2-1}\in\{0,k/2\}$, and from the second equation we have $a_{n/2}=-a_{n/2}$, i.e. $a_{n/2}\in\{0,k/2\}$; the result follows.

If n is odd then from the first equation we have $a_{(n-1)/2-1}=-a_{(n-1)/2-2}$, i.e. $c_0=-c_1$, and the result follows.

ii) By negasymmetry of $\mathbf{a}$, $a_i=-a_{n-2-i}$ for every i $(0\leq i\leq n-2)$, and by right-semi-negasymmetry of $\mathbf{b}$, $a_j=-a_{n+1-j}$ for every j $(2\leq j\leq n-1)$. Hence $a_i=a_{i+3}$, $0\leq i\leq n-3$. Now, since $n\geq 5$, $a_{3i+j}=c_j$, for every i and $0\leq j\leq 2$, for some c_j.

If $n\equiv 0 \pmod 3$ then, by negasymmetry, $c_0=a_0=-a_{n-2}=-c_1$, i.e. $c_0=-c_1$, and by right-semi-negasymmetry we have $c_2=a_2=-a_{n-1}=-c_2$, i.e. $c_2\in\{0,k/2\}$.

If $n\equiv 1 \pmod 3$ then, by negasymmetry, $c_0=a_0=-a_{n-2}=-c_2$ and so $c_0=-c_2$, and again by negasymmetry $c_1=a_1=-a_{n-3}=-c_1$, i.e. $c_1\in\{0,k/2\}$.

If $n\equiv 2 \pmod 3$ then, by negasymmetry, $c_1=a_1=-a_{n-3}=-c_2$, i.e. $c_1=-c_2$, and again by negasymmetry $c_0=a_0=-a_{n-2}=-c_0$, i.e. $c_0\in\{0,k/2\}$.

iii) By left-semi-negasymmetry of $\mathbf{a}$, $a_i = -a_{n-3-i}$ for every i $(0 \leq i \leq n-3)$, and by negasymmetry of $\mathbf{b}$, $a_j = -a_{n-j}$ for every j $(1 \leq j \leq n-1)$. Using exactly the same argument as for (ii), it follows that $a_{3i+j} = c_j$, for every i and $0 \leq j \leq 2$, for some c_j.
If $n \equiv 0 \pmod 3$ then, by negasymmetry, $c_1 = a_1 = -a_{n-1} = -c_2$ and so $c_1 = -c_2$, and by left-semi-negasymmetry $c_0 = a_0 = -a_{n-3} = -c_0$, i.e. $c_0 \in \{0, k/2\}$.
If $n \equiv 1 \pmod 3$ then, by negasymmetry, $c_1 = a_1 = -a_{n-1} = -c_0$ and so $c_1 = -c_0$, and again by negasymmetry $c_2 = a_2 = -a_{n-2} = -c_2$, i.e. $c_2 \in \{0, k/2\}$.
If $n \equiv 2 \pmod 3$ then, by left-semi-negasymmetry, $c_0 = a_0 = -a_{n-3} = -c_2$ and so $c_0 = -c_2$, and again by negasymmetry $c_1 = a_1 = -a_{n-1} = -c_1$, i.e. $c_1 \in \{0, k/2\}$.

iv) By left-semi-negasymmetry of $\mathbf{a}$, $a_i = -a_{n-3-i}$ for every i $(0 \leq i \leq n-3)$, and by right-semi-negasymmetry of $\mathbf{b}$, $a_j = -a_{n+1-j}$ for every j $(2 \leq j \leq n-1)$. Hence $a_i = a_{i+4}$, $0 \leq i \leq n-3$. Thus, since $n \geq 5$, $a_{4i+j} = c_j$, for every i and $0 \leq j \leq 3$, for some c_j.
If $n \equiv 0 \pmod 4$ then, by left-semi-negasymmetry, $c_0 = a_0 = -a_{n-3} = -c_1$, and so $c_0 = -c_1$, and by right-semi-negasymmetry, $c_2 = a_2 = -a_{n-1} = -c_3$, and so $c_2 = -c_3$.
If $n \equiv 1 \pmod 4$ then, by left-semi-negasymmetry, $c_0 = a_0 = -a_{n-3} = -c_2$, by left-semi-negasymmetry, $c_1 = a_1 = -a_{n-4} = -c_1$, i.e. $c_1 \in \{0, k/2\}$, and by right-semi-negasymmetry, $c_3 = a_3 = -a_{n-2} = -c_3$, and so $c_3 \in \{0, k/2\}$.
If $n \equiv 2 \pmod 4$ then, by left-semi-negasymmetry, $c_0 = a_0 = -a_{n-3} = -c_3$, and so $c_0 = -c_3$, and by right-semi-negasymmetry, $c_2 = a_2 = -a_{n-1} = -c_1$, and so $c_2 = -c_1$.
If $n \equiv 3 \pmod 4$ then, by left-semi-negasymmetry, $c_1 = a_1 = -a_{n-4} = -c_3$, by left-semi-negasymmetry, $c_0 = a_0 = -a_{n-3} = -c_0$, i.e. $c_0 \in \{0, k/2\}$, and by right-semi-negasymmetry, $c_2 = a_2 = -a_{n-1} = -c_2$, and so $c_2 \in \{0, k/2\}$.

□

3.4 Developing the Bound

Before establishing our new period bound, we first outline our strategy, enabling the proof of the bound to be described more simply.

In Sections 3.1 and 3.2 we described two ways of showing that certain edges in $B_k^-(n-1)$ cannot occur in $B^-(S,n)$ when S is a $\mathcal{NOS}_k(n)$. In particular we showed that in two cases incoming edges to certain categories of vertex cannot occur, and also in two cases that outgoing edges from certain categories of vertex cannot occur.

This suggests a straightforward strategy for bounding the period of an $\mathcal{NOS}_k(n)$, namely that the period is at most half the maximum cardinality of edges in $B^-(S,n)$ (by Lemma 3(i)). In turn, the number $|B^-(S,n)|$ of edges in $B^-(S,n)$ is bounded above by the number of edges in $B_k^-(n-1)$, i.e. $N_k(n)$, as specified in Lemma 2, less the number of edges in $B_k^-(n-1)$ that cannot occur in $B^-(S,n)$, as specified in Corollaries 2 and 3.

This strategy is complicated by the fact that, as noted in Section 3.3, there is a danger of 'double counting' certain excluded edges. For example, Corollary 2 asserts that certain outgoing edges from a left-sns $(n-1)$-tuple cannot occur, and Corollary 3 asserts that certain edges incoming to a uniform or negasymmetric $(n-1)$ tuple cannot occur. These two sets of excluded edges may overlap, and hence we need to take this into account; Lemma 9 is of key importance in this respect.

The following notation is intended to simplify the arguments. Let U_{in} and U_{out} be the sets of incoming and outgoing edges excluded by Corollary 2, and P_{in} and P_{out} be the sets of incoming and outgoing edges excluded by Corollary 3.

The above discussion leads to the following key lemma.

Lemma 10. *If $k > 2$ and S is an $\mathcal{NOS}_k(n)$ then:*

$$|B^-(S,n)| \leq \begin{cases} N_k(2) - |P_{out}|, & \text{if } n = 2 \\ N_k(3) - |P_{out}| - |P_{in}| + |P_{out} \cap P_{in}|, & \text{if } n = 3 \\ N_k(4) - |U_{out}| - |P_{out}|, & \text{if } n = 4 \\ N_k(n) - |U_{out}| - |U_{in}| - |P_{out}| - |P_{in}| & \\ \quad + |U_{out} \cap P_{in}| + |P_{out} \cap U_{in}| & \\ \quad + |U_{out} \cap U_{in}| + |P_{out} \cap P_{in}|, & \text{if } n > 4. \end{cases}$$

where $|X \cap Y|$ denotes the maximum possible cardinality for such a set, and $N_k(n)$ denotes the number of non-negasymmetric k-ary n-tuples (see Lemma 2).

Proof. The argument for $n = 2$ is immediate. A similar comment applies when $n = 3$. The $n = 4$ and $n \geq 5$ cases follow from observing that $U_{\text{in}} \cap P_{\text{in}} = U_{\text{out}} \cap P_{\text{out}} = \emptyset$ because a negasymmetric $(n-1)$-tuple can neither be non-uniform left-sns nor non-uniform right-sns, from Lemma 6. □

Remark 3. The reason to restrict the sets considered when $n \leq 4$ is because certain sets are empty or might be equal for small n. Moreover Corollary 2 only applies for $n \geq 4$.

3.5 The Bound

We can now give the main result.

Theorem 2. *Suppose $k > 2$ and $S = (s_i)$ is an $\mathcal{NOS}_k(n)$ of period m.*

i) If $n = 2$ then:

$$m \leq \begin{cases} \frac{k^2-k}{2}, & \text{if } k \text{ is odd} \\ \frac{k^2-k-2}{2}, & \text{if } k \text{ is even} \end{cases}$$

ii) If $n = 3$ then:

$$m \leq \begin{cases} \frac{k^3-3k+2}{2}, & \text{if } k \text{ is odd} \\ \frac{k^3-2k-6}{2}, & \text{if } k \text{ is even} \end{cases}$$

iii) If $n = 4$ then:

$$m \leq \begin{cases} \frac{k^4-2k^2+1}{2}, & \textit{if } k \textit{ is odd} \\ \frac{k^4-2k^2+k-2}{2}, & \textit{if } k \textit{ is even} \end{cases}$$

iv) If $n > 4$ then:

$$m \leq \begin{cases} \frac{k^n-5k^{(n-1)/2}+4k}{2}, & \textit{if } n \textit{ and } k \textit{ are odd} \\ \frac{k^n-6k^{(n-1)/2}+3k+2}{2}, & \textit{if } n \textit{ is odd and } k \textit{ is even} \\ \frac{k^n-3k^{n/2}-2k^{(n-2)/2}+k^2+3k}{2}, & \textit{if } n \textit{ is even and } k \textit{ is odd} \\ \frac{k^n-3k^{n/2}+k^2+k-2}{2}, & \textit{if } n \textit{ and } k \textit{ are even.} \end{cases}$$

Proof. The proof builds on, and uses the notation of, Lemma 10.

i) $n = 2$. If k is odd then $P_{\text{out}} = \emptyset$ by Corollary 3(i), since there are no negasymmetric non-uniform 1-tuples. If k is even then, by Corollary 3(iii), $|P_{\text{out}}| = 2$, since there are 2 negatively-alternating negasymmetric 1-tuples by Lemma 1(vi). The result follows from Lemma 10.

ii) $n = 3$. If k is odd then suppose S is a $\mathcal{NOS}_k(3)$. There are k negasymmetric 3-tuples, namely $(0, 0, 0)$ and $(s, 0, k-s)$, $1 \leq s \leq k-1$. Since the in-degree of every vertex in $B^-(S, 3)$ is the same as its out-degree, at least one of the 3-tuples corresponding to an incoming edge to the $k-1$ vertices $(s, 0)$, $1 \leq s \leq k-1$, cannot occur in $B^-(S, 3)$. Also, from Corollary 3(i), at least one of the 3-tuples corresponding to an incoming edge to the $k-1$ vertices $(s, k-s)$, $1 \leq s \leq k-1$, cannot occur in $B^-(S, 3)$. Thus at least $2(k-1)$ edges cannot occur. Put together with the k negasymmetric k-ary 3-tuples, this means that at least $3k-2$ edges cannot occur in $B^-(S, 3)$, and the result follows.

If k is even then, by Corollary 3(ii), $|P_{\text{in}}| = |P_{\text{out}}|$ equals the number of uniform or alternating negasymmetric 2-tuples where every element is either 0 or $k/2$, i.e. 4. Next observe that when evaluating $|P_{\text{in}} \cap P_{\text{out}}|$ we need only consider alternating 2-tuples in which every element is either 0 or $k/2$, since an edge can only be outgoing from a uniform negasymmetric $(n-1)$-tuple and incoming to a uniform negasymmetric $(n-1)$-tuple if it corresponds to a uniform negasymmetric n-tuple, and such an edge cannot occur in $B_k^-(n-1)$. Then $|P_{\text{in}} \cap P_{\text{out}}|$ is equal to the number of alternating negasymmetric n-tuples in which every element is either 0 or $k/2$, i.e. 2.

The result follows from Lemma 10.

iii) $n = 4$. By Corollary 2(v), $|U_{\text{out}}|$ equals the number of non-negatively-alternating left-sns 3-tuples, i.e. $k^2 - k$ by Lemma 1(x). If k is odd then, by Corollary 3(i), $|P_{\text{out}}|$ equals the number of negasymmetric non-uniform 3-tuples, i.e. $k-1$ by Lemma 1(i),(v). If k is even then, by Corollary 3(iii), $|P_{\text{out}}|$ equals the number of negatively-alternating negasymmetric 3-tuples, i.e. 2 by Lemma 1(vi). The result follows from Lemma 10.

iv)a) $n > 4$; **n and k odd**. By Corollary 2(i),(ii), $|U_{\text{out}}| = |U_{\text{in}}|$ equals the number of non-uniform left-sns $(n-1)$-tuples, i.e. $k^{(n-1)/2} - 1$ by Lemma 1(ix).

Also, by Lemma 9(iv), $|U_{\text{out}} \cap U_{\text{in}}| = k-1$ since if $n \equiv 1 \pmod 4$ there are k choices for c_0, and one choice each for c_1 and c_3, giving k possibilities, one of which is uniform (when $c_0 = c_1 = c_3 = 0$). An analogous argument applies if $n \equiv 3 \pmod 4$.

By Corollary 3 (i), $|P_{\text{out}}| = |P_{\text{in}}|$ equals the number of negasymmetric non-uniform $(n-1)$-tuples, i.e. $k^{(n-1)/2} - 1$ by Lemma 1(i),(v).

By Lemma 9(ii),(iii), $|U_{\text{out}} \cap P_{\text{in}}| = |P_{\text{out}} \cap U_{\text{in}}| = k-1$ (removing the case where the c_i are all equal).

By Lemma 9(i), $|P_{\text{out}} \cap P_{\text{in}}|$ equals the number of negasymmetric non-uniform $(n-1)$-tuples that are also negatively-alternating, i.e. $k-1$ by Lemma 1(iv),(vi).

The result follows from Lemma 10.

iv)b) $n > 4$**;** n **odd and** k **even**. By Corollary 2(iii),(iv), $|U_{\text{out}}| = |U_{\text{in}}|$ equals the number of non-uniform non-alternating left-sns $(n-1)$-tuples, i.e. $2k^{(n-1)/2} - 4$ by Lemma 1(xi).

Also, by Lemma 9(iv), $|U_{\text{out}} \cap U_{\text{in}}| = 4k-4$ since if $n \equiv 1 \pmod 4$ there are k choices for c_0, and two choices each for c_1 and c_3, giving $4k$ possibilities, two of which are uniform and two of which are alternating. An analogous argument applies if $n \equiv 3 \pmod 4$.

By Corollary 3(ii), $|P_{\text{out}}| = |P_{\text{in}}|$ equals the number of uniform or alternating negasymmetric $(n-1)$-tuples, i.e. k by Lemma 1(v),(vii).

By Corollary 3(ii), P_{out} and P_{in} only contain edges out-going/in-going from/to uniform or alternating $(n-1)$-tuples, and by Corollary 2(iii),(iv), U_{out} and U_{in} only contain edges that are out-going/in-going from/to non-uniform non-alternating $(n-1)$-tuples, and hence $|U_{\text{out}} \cap P_{\text{in}}| = |P_{\text{out}} \cap U_{\text{in}}| = \emptyset$.

By Corollary 3(ii), P_{out} and P_{in} only contain edges out-going/in-going from/to uniform or alternating negasymmetric $(n-1)$-tuples, which by Lemma 1(vii) are negatively-alternating. Hence an edge in $P_{\text{out}} \cap P_{\text{in}}$ must be negatively-alternating. An edge in $P_{\text{out}} \cap P_{\text{in}}$ cannot be negasymmetric, and there are $k-2$ non-negasymmetric negatively-alternating n-tuples by Lemma 1(iii),(v). Hence$|P_{\text{out}} \cap P_{\text{in}}| = k-2$.

The result follows from Lemma 10.

iv)c) $n > 4$**;** n **even**. By Corollary 2(v),(vi), $|U_{\text{out}}| = |U_{\text{in}}|$ equals the number of non-negatively-alternating left-sns $(n-1)$-tuples, i.e. $k^{n/2} - k$ by Lemma 1(x). Additionally, by Lemma 9(iv), $|U_{\text{out}} \cap U_{\text{in}}| = k(k-1)$, by choosing c_0 and c_2 to be distinct.

If k **is odd** then, by Corollary 3(i), $|P_{\text{out}}| = |P_{\text{in}}|$ equals the number of negasymmetric non-uniform $(n-1)$-tuples, i.e. $k^{(n-2)/2} - 1$ by Lemma 1(i),(v).

By Lemma 9(ii),(iii), $|U_{\text{out}} \cap P_{\text{in}}| = |P_{\text{out}} \cap U_{\text{in}}| = k-1$, by ensuring that c_0, c_1 and c_2 are not all equal to 0.

By Lemma 9(i), $|P_{\text{out}} \cap P_{\text{in}}| = 0$, i.e. the number of non-negatively-alternating $(n-1)$-tuples in which every entry is either 0 or $k/2$ when k is odd.

If k **is even** then, by Corollary 3(iii), $|P_{\text{out}}| = |P_{\text{in}}|$ equals the number of negatively-alternating negasymmetric $(n-1)$-tuples, i.e. 2 by Lemma 1(vi).

By Corollary 3(iii), P_{out} and P_{in} respectively only contain edges out-going from or in-going to negatively-alternating $(n-1)$-tuples, and by Corollary 2(v),(vi), U_{out} and U_{in} only contain edges that are out-going/in-going from/to non-negatively-alternating $(n-1)$-tuples, and hence $|U_{\mathrm{out}} \cap P_{\mathrm{in}}| = |P_{\mathrm{out}} \cap U_{\mathrm{in}}| = \emptyset$.

Finally, it is immediate that $|P_{\mathrm{out}} \cap P_{\mathrm{in}}| = 2$, i.e. the number of non-negatively-alternating $(n-1)$-tuples in which every entry is either 0 or $k/2$ when k is even.

The result follows from Lemma 10.

□

3.6 A Table

The bounds of Theorem 2 are tabulated for small k and n in Table 1. Note that the numbers given in brackets are computed from the bounds in Theorem 1, and are provided for comparison purposes.

Table 1. Bounds — new and (old) — on the period of an $\mathcal{NOS}_k(n)$

n	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
2	3	5	10	14	21	27	36
	(3)	(5)	(10)	(14)	(21)	(27)	(36)
3	10	25	56	99	162	245	352
	(11)	(27)	(58)	(101)	(165)	(247)	(356)
4	32	113	288	614	1152	1987	3200
	(35)	(119)	(298)	(629)	(1173)	(2015)	(3236)
5	105	471	1510	3790	8295	16205	29340
	(113)	(495)	(1538)	(3851)	(8355)	(16319)	(29444)
6	324	1961	7620	23024	58296	130339	264600
	(347)	(2015)	(7738)	(23219)	(58629)	(130815)	(265316)
7	1032	8007	38760	139330	410928	1047053	2389680
	(1067)	(8127)	(38938)	(139751)	(411429)	(1048063)	(2390756)
8	3141	32393	194270	837884	2878491	8382499	21512844
	(3227)	(32639)	(194938)	(839159)	(2881029)	(8386559)	(21519716)
9	9645	130311	975010	5034970	20170815	67096589	193693860
	(9761)	(130815)	(975938)	(5037551)	(20174403)	(67104767)	(193703684)

Observe that since, for k odd, $r_{k,3,3k/2} = 3k - 2$[1], and from [9, Lemma 3.9] there exists a $\mathcal{NOS}_k(n)$ with period $(k^n - r_{k,3,3k/2})/2$, Theorem 2(ii) implies that the sequences of [9, Lemma 3.9] have maximal period for $n = 3$ and odd k.

[1] Observing that $r_{k,3,3k/2}$ is the number of k-ary 3-tuples with pseudoweight exactly $3k/2$ then, since $3k$ is odd, a 3-tuple with pseudoweight exactly $3k/2$ must contain either one or three zeros. Thus the set of 3-tuples with pseudoweight $3k/2$ is $\{(0,0,0)\} \cup \{(s,k-s,0),(s,0,k-s),(0,s,k-s) : 1 \leq s \leq k-1\}$.

4 Concluding Remarks

Although for $n > 2$ the new bounds are sharper than those previously known, there remains a gap between the largest known period of a $\mathcal{NOS}_k(n)$ and the bound, even for small values of n. The current state of knowledge for small n and $k > 2$ is summarised in Table 2, where the upper bound from Theorem 2 is given in brackets beneath the largest known period of a $\mathcal{NOS}_k(n)$. The sequences with the largest known period are derived from [10, Theorem 4.9]. The values in bold represent maximal values.

Table 2. Largest known periods for an $\mathcal{NOS}_k(n)$ (and bounds)

n	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
2	**3**	**5**	**10**	**14**	**21**	**27**
	(3)	(5)	(10)	(14)	(21)	(27)
3	**10**	22	**56**	92	**162**	234
	(10)	(25)	(56)	(99)	(162)	(245)
4	31	99	284	591	1146	1955
	(32)	(113)	(288)	(614)	(1152)	(1987)
5	101	436	1502	3712	8283	16068
	(105)	(471)	(1510)	(3790)	(8295)	(16205)
6	316	1870	7596	22808	58248	129946
	(324)	(1961)	(7620)	(23024)	(58296)	(130339)
7	1002	7750	38628	138462	410574	1044998
	(1032)	(8007)	(38760)	(139330)	(410928)	(1047053)
8	3068	31751	193816	835495	2876916	8376327
	(3141)	(32393)	(194270)	(837884)	(2878491)	(8382499)

While it was already known [9, Note 3.12] how to construct optimal period negative orientable sequences for $n = 2$, and we have established that the negative orientable sequences of [9, Lemma 3.9] have optimal period for $n = 3$ and k odd, it should be clear that determining the optimal period for a $\mathcal{NOS}_k(n)$ remains an open question for $n = 3$ and k even, and all $n > 3$ regardless of k. Addressing this question remains an area for future research. Indeed, the smallest open cases (i.e. $n = 3$, $k = 4$ and $n = 4$, $k = 3$) appear readily resolvable by computer search.

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