Weight-constrained cut-down de Bruijn sequences

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7th August 2026 (v2)

Abstract

In 2011/12, Ruskey, Sawada, Stevens and Williams showed that a binary cut-down de Bruijn sequence can be constructed containing all n-tuples of Hamming weight either r or $r+1$ for every possible r, and no others. In this paper we examine extensions of this result that require choosing a weight-like function applying to n-tuples with symbols taken from an alphabet of arbitrary (finite) size. We consider three possible such weight functions, and in each case establish an analogous result to that of Ruskey et al.

1 Introduction

The main focus of this paper is a special class of *universal sequences* (or *universal cycles*), namely — at least for the purposes of this paper — infinite periodic sequences with elements from a finite set in which every n-tuple of a certain type occurs exactly once. Universal sequences of a range of types have been widely studied — see, for example, Chung et al. [7]. According to Hollmann and van Lint [11], the term universal sequence in this context dates back to a 1985 paper of Lempel and Cohn [14], although they use the term in a much more restricted sense. The more general definition we use here is consistent with the rather abstract definition given by Chroman et al. [6].

In this paper we are interested in sequences including all n-tuples for which some weight-like measure is restricted. Such sequences have been previously considered in the binary case by Sawada et al. [23] and Ruskey et al. [22], who considered binary n-tuples of restricted Hamming weight. In this paper we generalise the main existence result of Ruskey et al. to finite alphabets of arbitrary size. In doing so we look at three main examples of weight-like functions, one case of which was previously considered by Gabrić et al. [9].

1.1 Background

In this paper we consider periodic sequences (s_i) with elements from $\mathbb{Z}_k$ for some k, which we refer to as k-ary. For a sequence $S = (s_i)$ and $n \geq 1$ we write $\mathbf{s}_n(i) = (s_i, s_{i+1}, \ldots, s_{i+n-1})$, for an n-tuple corresponding to a substring or factor of n consecutive symbols occurring in the sequence at positions $i, i+1, \ldots, i+n-1$. We say such an n-tuple appears in the sequence.

Definition 1.1 ([2])**.** A k-ary *n-window sequence* $S = (s_i)$ is a periodic sequence of elements from $\mathbb{Z}_k$ ($k > 1$, $n > 1$) with the property that no n-tuple appears more than once in a period of the sequence, i.e. with the property that if $\mathbf{s}_n(i) = \mathbf{s}_n(j)$ for some i, j, then $i \equiv j \pmod{m}$ where m is the period of the sequence. Such a sequence is also known as a *cut-down de Bruijn sequence* (see, for example, Cameron et al. [5]), and this is the term we use here.

A k-ary de Bruijn sequence *of span* n is then simply an n-window sequence in which every k-ary n-tuple appears once in a period, i.e. an n-window sequence of maximal period.

Following Alhakim et al. [2] and many previous authors, we also introduce the k-ary de Bruijn digraph. For positive integers n and k greater than one, let $\mathbb{Z}_k^n$ be the set of all k^n tuples of length n with entries from the group $\mathbb{Z}_k$ of residues modulo k. The order n de Bruijn digraph, $B_k(n)$, is a directed graph with $\mathbb{Z}_k^n$ as its vertex set in which, for any two vertices $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$ and $\mathbf{y} = (y_0, y_1, \ldots, y_{n-1})$, the pair $(\mathbf{x}, \mathbf{y})$ is an edge if and only if $y_i = x_{i+1}$ for every i ($0 \leq i < n-1$). We label such an edge with the $(n+1)$-tuple $(x_0, x_1, \ldots, x_{n-1}, y_{n-1})$.

Note that we have defined two ways of specifying an edge in $B_k(n)$, namely as either a pair of vertices $(\mathbf{a}, \mathbf{b})$, where $\mathbf{a}, \mathbf{b}$ are k-ary n-tuples, or as a single k-ary $(n+1)$-tuple $\mathbf{x}$. We refer to an edge in $B_k(n)$ and a k-ary n-tuple interchangeably throughout.

Definition 1.2. An *Eulerian digraph* is a strongly connected digraph, i.e. one in which there is a directed path between any pair of vertices, for which every vertex has in-degree equal to out-degree. This latter property characterises a *balanced* digraph.

The name derives from the fact that there exists an Eulerian circuit, i.e. a directed path visiting every edge once, in a digraph if and only if the digraph is Eulerian — see, for example, Corollary 6.1 of Gibbons [10] or Knuth [13, Section 2.3.4.2, Theorem G]. Moreover, there are simple and efficient algorithms for finding Eulerian circuits — see for example [10, Figure 6.5] or [13, Section 2.3.4.2, Theorem D].

There is then a simple correspondence between a directed Eulerian circuit in $B_k(n-1)$ and a de Bruijn sequence of span n; this generalises in an obvious way so that a directed circuit (or cycle) in $B_k(n-1)$ of length $m \leq k^n$ corresponds to a cut-down de Bruijn sequence of period m.

We also need the following.

Definition 1.3 (Necklace — see, for example, [21])**.** A k-ary *necklace* within the de Bruijn graph $B_k(n-1)$, written as $[a_0, a_1, \ldots, a_{m-1}]$, where $m \mid n$, consists of all distinct n-tuples of the form $(a_{\overline{i}}, a_{\overline{i+1}}, \ldots, a_{\overline{n-1}}, a_0, a_1, \ldots, a_{\overline{i-1}})$, for $0 \leq i < m$, i.e. all cyclic shifts, where $\overline{s} = s \bmod m$.

De Bruijn sequences have a range of real-world applications, ranging from position location on a linear track or rotating shaft [3, 4, 18, 20, 25], to genome assembly [8]. As observed by Cameron et al. [5], 'for some applications it may be more convenient to produce a cycle of arbitrary length such that there are no repeated strings', i.e. a cut-down de Bruijn sequence of a particular period that derives from the specifics of the application.

A number of authors have considered the problem of constructing cut-down de Bruijn sequences with particular properties. Jackson et al. [12, Problem 477] observed that a binary n-window sequence exists for every possible period m satisfying $n \leq m \leq 2^n$. Ruskey et al. [22] showed that a cut-down de Bruijn sequence can be constructed containing all n-tuples with weights in the range $[a, b]$ as long as $0 \leq a < b \leq n$, and such sequences were further studied by Li et al. [15].

Sequences containing only n-tuples with specified weights are a particular example of what are sometimes known as 'de Bruijn sequences with forbidden subsequences', as for example examined in the binary case by Penne [19] and for arbitrary alphabets by Alhakim [1]. Tan and Shallot [24], studied the more general problem of the existence of sequences containing only the members of a pre-specified set of n-tuples, where they allow such n-tuples to occur multiple times in a period. These are all examples of universal sequences, as introduced at the beginning of this paper.

1.2 Cut down de Bruijn sequences in weight-restricted subgraphs

As observed at the start of this paper, we are concerned with a particular class of cut-down de Bruijn sequences, as previously considered by Sawada et al. [23] and Ruskey et al. [22]. They showed that, if $n \geq 1$ and $0 \leq r < n$, the sub-graph of the de Bruijn graph $B_2(n)$ consisting of all edges corresponding to n-tuples of Hamming weights r and $r+1$ is Eulerian, and hence there exists a cut down de Bruijn sequence which includes all n-tuples of weights r and $r+1$ (and no others).

In this note we consider to what extent a similar result holds for the k-ary de Bruijn graph. Of course, in this case there are a number of ways of computing the 'weight' of an n-tuple, and we therefore consider what happens for three different functions that assign a weight-like value to an n-tuple.

2 Weight functions for k-ary alphabets

There are many ways of generalising the notion of Hamming weight of a binary n-tuple to the k-ary case. We briefly introduce some key possibilities here. The 'standard' generalisation of Hamming weight to the k-ary case is as follows.

Definition 2.1 (Hamming weight)**.** Suppose $k \geq 2$ and $n \geq 1$. If $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ is a k-ary n-tuple, then the *Hamming weight* $w_h(\mathbf{a})$ of $\mathbf{a}$ is equal to the number of values i for which $a_i \neq 0$.

It should be clear that in the binary case, i.e. where $k = 2$, the Hamming weight corresponds to the number of ones in the n-tuple, i.e. the expected definition. An alternative measure for n-tuples, which also corresponds to Hamming weight in the case $k = 2$, is as follows.

Definition 2.2 (Digit sum)**.** Suppose $k \geq 2$, $n \geq 1$ and $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ is a k-ary n-tuple. Then the *digit sum* (or *symbol sum*) of $\mathbf{a}$ is defined as

$$w_s(\mathbf{a}) = \sum_{i=0}^{n-1} a_i$$

where we compute the sum in $\mathbb{Z}$ and treat the values a_i as integers in the range $[0, k-1]$.

Remark 2.1. Note that the digit sum is referred to simply as the *weight* of an n-tuple in [16, Definition 11]. It should be clear that the digit sum is also a natural generalisation of Hamming weight for the binary case, although the digit sum does not have the properties one would normally expect of a distance function.

Yet another alternative way of assigning a numeric value to an n-tuple is as follows.

Definition 2.3 ([16], Definition 10)**.** Suppose $k \geq 2$, $n \geq 1$ and $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ is a k-ary n-tuple. Define the function $f : \mathbb{Z}_k \rightarrow \mathbb{Q}$ as follows: for any $a \in \mathbb{Z}_k$ treat a as an integer in the range $[0, k-1]$ and set

$f(a) = a$ if $a \neq 0$ and $f(a) = k/2$ if $a = 0$. Then the *pseudoweight* of $\mathbf{a}$ is defined to be the sum

$$w_p(\mathbf{a}) = \sum_{i=0}^{n-1} f(a_i)$$

where the sum is computed in $\mathbb{Q}$.

This apparently curious notion was introduced to help generate special classes of cut-down de Bruijn sequences known as orientable sequences and negative orientable sequences (see, for example, [17]).

In the remainder of this paper we consider to what extent generalised versions of the Ruskey et al. result [22] hold for these three weight functions. We first need the following preliminary definition and result that simplify arguments for all three weight functions we examine.

Definition 2.4. Suppose $k \geq 2$, $n \geq 1$ and w is function mapping k-ary n-tuples to $\mathbb{Q}$. Then w is said to be *permutation-agnostic* if, for any k-ary n-tuple $\mathbf{a}$, $w(\mathbf{a}) = w(p(\mathbf{a}))$ for every permutation p acting on the n entries in $\mathbf{a}$.

The three weight functions w_h, w_s and w_p are all permutation-agnostic since they are all defined as a function of the elements of an n-tuple in a way that does not depend on the ordering.

Lemma 2.1. *Suppose $k \geq 2$, $n \geq 2$, $r \leq s \in \mathbb{Q}$, and w is permutation-agnostic function mapping kary n-tuples to $\mathbb{Q}$. Let $W_k^{r,s}(n-1)$ be the subgraph of $B_k(n-1)$ consisting of the edges corresponding to n-tuples $\mathbf{a}$ satisfying $r \leq w(\mathbf{a}) \leq s$. Then, if $\mathbf{a}$ and $\mathbf{b}$ are edges (n-tuples)) in $W_k^{r,s}(n-1)$ that satisfy $w(\mathbf{a}) = r$ and $w(\mathbf{b}) = s$ and that differ in a single position, then there are directed paths from $\mathbf{a}$ to $\mathbf{b}$ and from $\mathbf{b}$ to $\mathbf{a}$ in $W_k^{r,s}(n-1)$.*

Proof. Suppose $\mathbf{a} = (a_0, a_1 \ldots, a_{n-1})$ and $\mathbf{b} = (b_0, b_1 \ldots, b_{n-1})$, where $a_i = b_i$ for every i except $i = s$ for some s ($0 \leq s \leq n-1$). Since w is permutation-agnostic, if $\mathbf{a}'$ is a cyclic shift of $\mathbf{a}$ then $w(\mathbf{a}) = w(\mathbf{a}')$. Hence there is a directed path from $\mathbf{a}$ to $(a_s, a_{s+1}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{s-1})$ in $W_k^{r,s}(n-1)$. Clearly $(a_s, a_{s+1}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{s-1})$ is an incoming edge to the vertex $(a_{s+1}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{s-1})$ and $(a_{s+1}, a_{s+2}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{s-1}, b_s) = (b_{s+1}, b_{s+2}, \ldots, b_{n-1}, b_0, b_1, \ldots, b_s)$ is an outgoing edge from this vertex. That is, we can extend the directed path so that it goes from $\mathbf{a}$ to a cyclic shift of $\mathbf{b}$. Using the same argument as previously, there is a directed path from $(b_{s+1}, b_{s+2}, \ldots, b_{n-1}, b_0, b_1, \ldots, b_s)$ to $\mathbf{b}$ in $W_k^{r,s}(n-1)$. The same argument works in reverse, and the result follows. $\square$

3 Partitioning using Hamming weight

We start by considering the case where the weight of an n-tuple is simply the number of non-zero entries, i.e. using the function w_h as specified in Definition 2.1. We have a result very similar to that of Ruskey et al. [22]. We first need the following notation.

Definition 3.1 (The Hamming weight subgraph)**.** Suppose $k \geq 2$, $n \geq 1$ and $0 \leq r \leq s \leq n$. We write $H_k^{r,s}(n-1)$ for the subgraph of $B_k(n-1)$ consisting of the edges corresponding to n-tuples $\mathbf{a}$ satisfying $r \leq w_h(\mathbf{a}) \leq s$.

Remark 3.1. $H_2^{r-1,r}(n-1)$ is what Ruskey et al. [22] refer to as $G(\mathbf{B}_{r-1}^r(n-1))$.

The following two results are straightforward to establish.

Lemma 3.1. *If $k \geq 2$, $n \geq 2$ and $0 \leq r \leq s \leq n$ then $H_k^{r,s}(n-1)$ is balanced.*

Proof. $H_k^{r,r}(n-1)$ is balanced for any r, since if $(a_1, a_2, \ldots, a_{n-1})$ is a vertex in $B_k(n-1)$ and $x \in \mathbb{Z}_k$, then $(x, a_1, a_2, \ldots, a_{n-1})$ is an incoming edge in $H_k^{r,r}(n-1)$ if and only if $(a_1, a_2, \ldots, a_{n-1}, x)$ is an outgoing edge in $H_k^{r,r}(n-1)$. But the set of edges in $H_k^{r,s}(n-1)$ is simply the union of the sets of edges in $H_k^{t,t}(n-1)$ for every t satisfying $r \leq t \leq s$, and the result follows. $\square$

Lemma 3.2. *Suppose $k \geq 2$, $n \geq 2$, $0 \leq r \leq n$ and $(a_1, a_2, \ldots, a_{n-1})$ is a vertex in $H_k^{r,r}(n-1)$. If $(a_0, a_1, \ldots, a_{n-1})$ is an incoming edge to this vertex then $(a_1, a_2, \ldots, a_n)$ is an outgoing edge if and only if either $a_0 = a_n = 0$ or a_0 and a_n are both non-zero.*

Proof. This follows since all edges in $H_k^{r,r}(n-1)$ must correspond to n-tuples with precisely r non-zero entries. $\square$

Before giving the main result we need the following key lemma.

Lemma 3.3. *Suppose $n \geq 2$, $k \geq 2$ and $0 \leq r < n$.*

(i) If $\mathbf{a}$ is an edge (i.e. an n-tuple) in $H_k^{r,r+1}(n-1)$, then there is a directed path in $H_k^{r,r+1}(n-1)$ from $\mathbf{a}$ to an edge $\mathbf{a}'$ satisfying $w_h(\mathbf{a}') = r$.

(ii) If $\mathbf{b}$ is an edge (i.e. an n-tuple) in $H_k^{r,r+1}(n-1)$, then there is a directed path in $H_k^{r,r+1}(n-1)$ from an edge $\mathbf{b}'$ satisfying $w_h(\mathbf{b}') = r+1$ to $\mathbf{b}$.

(iii) *If* $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ *is an edge (i.e. an* n*-tuple) in* $H_k^{r,r+1}(n-1)$ *such that* $w_h(\mathbf{a}) = r$ *and* $a_i = 0$ *for some* i*, then there is a directed path in* $H_k^{r,r+1}(n-1)$ *from* $\mathbf{a}$ *to*

$$(a_0, a_1, \ldots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \ldots, a_{n-1}),$$

i.e. $\mathbf{a}$ *modified so that* a_i *and* a_{i+1} *are interchanged.*

(iv) *If* $\mathbf{b} = (b_0, b_1, \ldots, b_{n-1})$ *is an edge (i.e. an* n*-tuple) in* $H_k^{r,r+1}(n-1)$ *such that* $w_h(\mathbf{b}) = r+1$ *and* $b_i = 0$ *for some* i*, then there is a directed path in* $H_k^{r,r+1}(n-1)$ *from*

$$(b_0, b_1, \ldots, b_{i-1}, b_{i+1}, b_i, b_{i+2}, \ldots, b_{n-1})$$

i.e. $\mathbf{b}$ *modified so that* b_i *and* b_{i+1} *are interchanged, to* $\mathbf{b}$*.*

Proof. (i) If $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ satisfies $w_h(\mathbf{a}) = r+1$, then suppose $a_i \neq 0$ for some i (which must exist since $w_h(\mathbf{a}) > 0$). Now clearly there is a directed path in $H_k^{r,r+1}(n-1)$ from $\mathbf{a}$ to $(a_i, a_{i+1}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{i-1})$, simply by taking successive cyclic shifts. Because $w_h(\mathbf{a}) = r+1$ and $a_i \neq 0$, the edge $(a_{i+1}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{i-1}, 0)$ is an edge in $H_k^{r,r+1}(n-1)$ with weight r, and so a directed path exists in $H_k^{r,r+1}(n-1)$ from $\mathbf{a}$ to an edge with weight r. Alternatively, if $w_h(\mathbf{a}) = r$, then simply set $\mathbf{a} = \mathbf{a}'$. Either way the result holds.

(ii) If $\mathbf{b} = (b_0, b_1, \ldots, b_{n-1})$ satisfies $w_h(\mathbf{b}) = r < n$, then there is at least one value b_i such that $b_i = 0$. As in the previous case, there is a directed path in $H_k^{r,r+1}(n-1)$ from $(b_i, b_{i+1}, \ldots, b_{n-1}, b_0, b_1, \ldots, b_{i-1})$ to $\mathbf{b}$. Because $w_h(\mathbf{b}) = r$, the edge $(1, b_{i+1}, \ldots, b_{n-1}, b_0, b_1, \ldots, b_{i-1})$ is an edge in $H_k^{r,r+1}(n-1)$ with Hamming weight $r+1$, and the desired directed path exists. Alternatively, if $w_h(\mathbf{b}) = r+1$, then simply set $\mathbf{b} = \mathbf{b}'$. Either way the result holds.

(iii) There is clearly a directed path from $\mathbf{a}$ to $(a_i, a_{i+1}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{i-1})$ in $H_k^{r,r+1}(n-1)$, simply by taking successive cyclic shifts. Since $\mathbf{a}$ has weight r and $a_i = 0$, the n-tuple

$$(a_{i+1}, a_{i+2}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{i-1}, a_{i+1})$$

has weight r or $r+1$ and hence is in $H_k^{r,r+1}(n-1)$. Clearly

$$(a_{i+2}, a_{i+3}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{i-1}, a_{i+1}, a_i)$$

has weight the same as $\mathbf{a}$ and hence is also in $H_k^{r,r+1}(n-1)$. That is, the desired directed path exists.

(iv) There is a directed path from $(b_{i+2}, b_{i+3}, \ldots, b_{n-1}, b_0, b_1, \ldots, b_{i+1})$ to $\mathbf{b}$ in $H_k^{r,r+1}(n-1)$, taking cyclic shifts. Because $\mathbf{b}$ has weight $r+1$ and $b_i = 0$, the n-tuple

$$(b_i, b_{i+2}, \ldots, b_{n-1}, b_0, b_1, \ldots, b_i)$$

has weight r or $r+1$ and hence is in $H_k^{r,r+1}(n-1)$. Also

$$(b_{i+1}, b_i, b_{i+2}, \ldots, b_{n-1}, b_0, b_1, \ldots, b_{i-1})$$

has weight the same as $\mathbf{b}$ and so is likewise in $H_k^{r,r+1}(n-1)$. This completes the proof.

□

We can now give our main result.

Theorem 3.4. *Suppose $k \geq 2$ and $n \geq 2$. Then:*

(i) $H_k^{r,r}(n-1)$ is Eulerian if and only if $r \in \{0, 1, n-1, n\}$;

(ii) $H_k^{r,r+1}(n-1)$ is Eulerian for every r $(0 \leq r < n)$.

Remark 3.2. For the case $k = 2$ this is precisely [22, Theorem 2.4].

Observe that in the remainder of the paper we write x^m as a shorthand for m consecutive repetitions of the symbol x.

Proof. Because of Lemma 3.1, establishing the Eulerian property just requires us to show that the subgraphs are strongly connected.

(i) The result is trivially true for $r = 0$, since there is a single directed edge with Hamming weight 0, the loop to and from the vertex 0^{n-1}. If $r = n$, then the edges in $H_k^{n,n}(n-1)$ are all the n-tuples that do not contain a zero. But this subgraph is isomorphic to the full de Bruijn graph $B_k(n-2)$, and hence it is clearly strongly connected.

Now suppose $r = 1$. The edges of $H_k^{1,1}(n-1)$ correspond to all n-tuples containing a single non-zero entry. Suppose $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ and $\mathbf{b} = (b_0, b_1, \ldots, b_{n-1})$ are two such n-tuples, where $a_u \neq 0$ and $b_v \neq 0$ and all the other a_i and b_i values are zero. Then there is clearly a directed path in $H_k^{1,1}(n-1)$ from $\mathbf{a}$ to the n-tuple $(a_u, 0, 0, \ldots, 0)$ involving taking successive cyclic shifts of $\mathbf{a}$, and similarly there is also a directed path from the n-tuple $(0, 0, \ldots, 0, b_v)$ to $\mathbf{b}$. Also, $(a_u, 0, 0, \ldots, 0)$ is an incoming edge to the vertex 0^{n-1} and $(0, 0, \ldots, 0, b_v)$ is an outgoing edge from the same vertex. Hence there is a directed path from $\mathbf{a}$ to $\mathbf{b}$ and we have established strong connectivity.

Finally, a somewhat similar argument applies when $r = n-1$. Suppose $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ and $\mathbf{b} = (b_0, b_1, \ldots, b_{n-1})$ are two n-tuples corresponding to edges in $H_k^{n-1,n-1}(n-1)$, where $a_u = b_v = 0$ and all the other a_i and b_i values are non-zero. Then there are directed paths in $H_k^{n-1,n-1}(n-1)$:

- from $\mathbf{a}$ to the n-tuple $(a_{u+1}, a_{u+2}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{u-1}, 0)$, just by taking cyclic shifts;
- then from $(a_{u+1}, a_{u+2}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{u-1}, 0)$ to $(0, b_{v+1}, b_{v+2}, \ldots, b_{n-1}, b_0, b_1, \ldots, b_{v-1})$ (from repeated application of Lemma 3.2); and
- finally from $(0, b_{v+1}, b_{v+2}, \ldots, b_{n-1}, b_0, b_1, \ldots, b_{v-1})$ to $\mathbf{b}$, again taking cyclic shifts;

yielding the required directed path from $\mathbf{a}$ to $\mathbf{b}$.

Finally suppose $1 < r < n-1$, and thus $n \geq 4$ — we need to show that $H_k^{r,r}(n-1)$ is not strongly connected. Regardless of r there will exist two edges in $H_k^{r,r}(n-1)$ corresponding to the n-tuples $\mathbf{a} = (1^r, 0^{n-r})$ and $\mathbf{b} = (1, 0, 1^{r-1}, 0^{n-r-1})$, and all the constituent strings are non-empty since $n \geq 4$ and $1 < r < n-1$. We claim that there is no path in $H_k^{r,r}(n-1)$ linking $\mathbf{a}$ to $\mathbf{b}$.

This holds because, from Lemma 3.2, any edge on a path in $H_k^{r,r}(n-1)$ starting from $\mathbf{a}$ must contain r consecutive non-zero entries (working cyclically). Similarly, any edges on a path starting from $\mathbf{b}$ must contain a zero followed by $r-1$ consecutive non-zero entries and then another zero (again working cyclically). That is $\mathbf{b}$ cannot be on a path starting at $\mathbf{a}$, and vice versa.

(ii) Suppose $\mathbf{a}$ and $\mathbf{b}$ are vertices in $H_k^{r,r+1}(n-1)$. Then, by Lemma 3.3(i) and (ii), there are directed paths in $H_k^{r,r+1}(n-1)$ from $\mathbf{a}$ to $\mathbf{a}'$ and from $\mathbf{b}'$ to $\mathbf{b}$, where $\mathbf{a}'$ has weight r and $\mathbf{b}'$ has weight $r+1$. So to establish the result we simply need to show there is a path in $H_k^{r,r+1}(n-1)$ from any n-tuple with weight r to any n-tuple with weight $r+1$.

Suppose $\mathbf{a}$ has weight r and $\mathbf{b}$ has weight $r+1$, where $0 \leq r < n$. Then we claim there are directed paths in $H_k^{r,r+1}(n-1)$ from $\mathbf{a}$ to $(1^r, 0^{n-r})$, and from $(1^{r+1}, 0^{n-r-1})$ to $\mathbf{b}$. The first assertion follows first from repeated application of Lemma 3.3(iii) to $\mathbf{a}$ to give a path from $\mathbf{a}$ to an edge whose first r entries are non-zero and the last $n-r$ entries are zero. A path from this vertex to $(1^r, 0^{n-r})$ exists from Lemma 3.2.

An exactly analogous argument using repeated applications of Lemmas 3.3(iv) and 3.2 shows that there is a directed path from $(1^{r+1}, 0^{n-r-1})$ to $\mathbf{b}$. That is, to complete the proof it remains to show that there is a directed path in $H_k^{r,r+1}(n-1)$ from $(1^r, 0^{n-r})$ to $(1^{r+1}, 0^{n-r-1})$; but this is immediate since they differ in only one digit and hence the result follows from Lemma 2.1.

□

Remark 3.3. A simpler somewhat less formal approach to proving part (i) of the above theorem is to observe that the n-tuples corresponding to edges in $H_k^{r,r}(n-1)$ all have the same 'pattern' of occurrences of zeros, up to cyclic shift. This follows from Lemma 3.2.

Theorem 3.4 has the following immediate corollary.

Corollary 3.5. *Suppose $k \geq 2$, $n \geq 2$ and $0 \leq r < s \leq n$. Then $H_k^{r,s}(n-1)$ is Eulerian.*

Proof. The fact that $H_k^{r,s}(n-1)$ is balanced is immediate from Lemma 3.1. We know that the set of edges in $H_k^{r,s}(n-1)$ is the union of the sets of edges in $H_k^{t,t+1}(n-1)$ for every t satisfying $r \leq t < s$, and $H_k^{t,t+1}(n-1)$ is strongly connected from theorem 3.4(ii). So suppose $\mathbf{a}_t$ and $\mathbf{a}_u$ are edges in $H_k^{r,s}(n-1)$ of weights t and u respectively. Without loss of generality suppose $t \leq u$. If $t = u$ or $t = u+1$ then both $\mathbf{a}_t$ and $\mathbf{a}_u$ are in $H_k^{t,t+1}(n-1)$, and hence there is a directed path from $\mathbf{a}_t$ to $\mathbf{a}_u$. So suppose $u > t+1$. Then let $\mathbf{a}_{t+1}, \mathbf{a}_{t+2}, \ldots, \mathbf{a}_{u-1}$ be arbitrary edges of weights $t+1, t+2, \ldots, u-1$ respectively. Then since $H_k^{i,i+1}(n-1)$ is strongly connected for every i, there exist directed paths in $H_k^{r,s}(n-1)$ from $\mathbf{a}_i$ to $\mathbf{a}_{i+1}$ for every i ($u \leq i < t$), and hence from $\mathbf{a}_t$ to $\mathbf{a}_u$. It follows that $H_k^{r,s}(n-1)$ is strongly connected. □

4 Partitioning using digit sum

The second 'measure' for n-tuples that we consider is the digit sum, as in Definition 2.2. For this case we need the following.

Definition 4.1 (The digit sum subgraph)**.** Suppose $k \geq 2$, $n \geq 2$ and $0 \leq r \leq s \leq n(k-1)$. We write $S_k^{r,s}(n-1)$ for the subgraph of $B_k(n-1)$ consisting of the edges corresponding to n-tuples $\mathbf{a}$ satisfying $r \leq w_s(\mathbf{a}) \leq s$.

A special case of the problem we address here has been previously considered for this function.

Result 4.1 ([9])**.** *Suppose $k \geq 2$ and $n \geq 2$. Then $S_k^{0,s}(n-1)$ and $S_k^{s,n(k-1)}(n-1)$ are Eulerian, for every s ($0 \leq s \leq n(k-1)$).*

In fact a much more general result can be established. Analogously to Lemmas 3.1 and 3.2 we first have the following.

Lemma 4.2. *If $k \geq 2$, $n \geq 2$ and $0 \leq r \leq s \leq n(k-1)$ then $S_k^{r,s}(n-1)$ is balanced.*

Proof. The fact that $S_k^{r,r}(n-1)$ is balanced for any r is immediate, since if $(a_1, a_2, \ldots, a_{n-1})$ is a vertex in $B_k(n-1)$ and $x \in \mathbb{Z}_k$, then $(x, a_1, a_2, \ldots, a_{n-1})$ is an incoming edge in $S_k^{r,r}(n-1)$ if and only if $(a_1, a_2, \ldots, a_{n-1}, x)$ is an outgoing edge in $S_k^{r,r}(n-1)$. But the set of edges in $S_k^{r,s}(n-1)$ is simply the union of the sets of edges in $S_k^{t,t}(n-1)$ for every t satisfying $r \leq t \leq s$, and the result follows. $\square$

Lemma 4.3. *Suppose $k \geq 2$, $n \geq 2$, $0 \leq r \leq n(k-1)$ and $(a_1, a_2, \ldots, a_{n-1})$ is a vertex in $S_k^{r,r}(n-1)$. If $(a_0, a_1, \ldots, a_{n-1})$ is an incoming edge to this vertex then $(a_1, a_2, \ldots, a_n)$ is an outgoing edge if and only if $a_0 = a_n$.*

Proof. This is immediate since all edges in $S_k^{r,r}(n-1)$ correspond to n-tuples with digit sum r. $\square$

Since $w_s(\mathbf{a}) = w_h(\mathbf{a})$ if $k = 2$, Theorem 3.4 applies in this case. Thus we restrict our attention here to the case $k > 2$. Before giving the main result we need the following key lemma.

Lemma 4.4. *Suppose $n \geq 2$, $k > 2$ and $0 \leq r < n(k-1)$.*

(i) If $\mathbf{a}$ is an edge (i.e. an n-tuple) in $S_k^{r,r+1}(n-1)$, then there is a directed path in $S_k^{r,r+1}(n-1)$ from $\mathbf{a}$ to an edge $\mathbf{a}'$ satisfying $w_s(\mathbf{a}') = r$.

(ii) If $\mathbf{b}$ is an edge (i.e. an n-tuple) in $S_k^{r,r+1}(n-1)$, then there is a directed path in $S_k^{r,r+1}(n-1)$ from an edge $\mathbf{b}'$ satisfying $w_s(\mathbf{b}') = r+1$ to $\mathbf{b}$.

(iii) If $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ is an edge (i.e. an n-tuple) in $S_k^{r,r+1}(n-1)$ such that $w_s(\mathbf{a}) = r$, $a_i < k-1$ and $a_{i+1} > 0$, for some i, then there is a directed path in $S_k^{r,r+1}(n-1)$ from $\mathbf{a}$ to

$$(a_0, a_1, \ldots, a_{i-1}, a_i + 1, a_{i+1} - 1, a_{i+2}, \ldots, a_{n-1}),$$

i.e. $\mathbf{a}$ modified so that a_i is increased by 1 and a_{i+1} decreased by 1.

(iv) If $\mathbf{b} = (b_0, b_1, \ldots, b_{n-1})$ is an edge (i.e. an n-tuple) in $S_k^{r,r+1}(n-1)$ such that $w_s(\mathbf{b}) = r+1$, $b_i < k-1$ and $b_{i+1} > 0$, for some i, then there is a directed path in $S_k^{r,r+1}(n-1)$ from

$$(b_0, b_1, \ldots, b_{i-1}, b_i + 1, b_{i+1} - 1, b_{i+2}, \ldots, b_{n-1})$$

i.e. $\mathbf{b}$ modified so that b_i is increased by 1 and b_{i+1} decreased by 1, to $\mathbf{b}$.

Proof. (i) If $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ satisfies $w_s(\mathbf{a}) = r + 1$, then suppose $a_i \neq 0$ for some i (which must exist since $w_s(\mathbf{a}) > 0$). Now using cyclic shifting there is a directed path in $S_k^{r,r+1}(n-1)$ from $\mathbf{a}$ to $(a_i, a_{i+1}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{i-1})$. Because $w_s(\mathbf{a}) = r + 1$, the edge $(a_{i+1}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{i-1}, a_i - 1)$ is an edge in $S_k^{r,r+1}(n-1)$ with digit sum r, and so a directed path exists in $S_k^{r,r+1}(n-1)$ from $\mathbf{a}$ to an edge with digit sum r. Alternatively, if $w_s(\mathbf{a}) = r$, then simply set $\mathbf{a} = \mathbf{a}'$. Either way the result holds.

(ii) If $\mathbf{b} = (b_0, b_1, \ldots, b_{n-1})$ satisfies $w_s(\mathbf{b}) = r < n(k-1)$, then there is at least one value b_i such that $b_i < k - 1$. As in the previous case, using cyclic shifting there is a directed path in $S_k^{r,r+1}(n-1)$ from $(b_i, b_{i+1}, \ldots, b_{n-1}, b_0, b_1, \ldots, b_{i-1})$ to $\mathbf{b}$. Because $w_s(\mathbf{b}) = r$, the edge $(b_i + 1, b_{i+1}, \ldots, b_{n-1}, b_0, b_1, \ldots, b_{i-1})$ is an edge in $S_k^{r,r+1}(n-1)$ with digit sum $r + 1$, and the desired directed path exists. Alternatively, if $w_s(\mathbf{b}) = r + 1$, then simply set $\mathbf{b} = \mathbf{b}'$. Either way the result holds.

(iii) There is clearly a directed path from $\mathbf{a}$ to $(a_i, a_{i+1}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{i-1})$ in $S_k^{r,r+1}(n-1)$ simply from cyclic shifting. Since $\mathbf{a}$ has digit sum r, the n-tuple

$$(a_{i+1}, a_{i+2}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{i-1}, a_i + 1)$$

has digit sum $r + 1$ and hence is in $S_k^{r,r+1}(n-1)$. It follows that

$$(a_{i+2}, a_{i+3}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{i-1}, a_i + 1, a_{i+1} - 1)$$

has digit sum r and hence is also in $S_k^{r,r+1}(n-1)$. Finally, there is a directed path from this latter n-tuple to

$$(a_0, a_1, \ldots, a_{i-1}, a_i + 1, a_{i+1} - 1, a_{i+2}, \ldots, a_{n-1}),$$

via cyclic shifting and so the desired directed path exists.

(iv) There is a directed path from $(b_{i+2}, b_{i+3}, \ldots, b_{n-1}, b_0, b_1, \ldots, b_{i+1})$ to $\mathbf{b}$ in $S_k^{r,r+1}(n-1)$ using cyclic shifting. Because $\mathbf{b}$ has digit sum $r+1$, the n-tuple

$$(b_{i+1} - 1, b_{i+2}, \ldots, b_{n-1}, b_0, b_1, \ldots, b_i)$$

has digit sum r and hence is in $S_k^{r,r+1}(n-1)$. Hence

$$(b_i + 1, b_{i+1} - 1, b_{i+2}, \ldots, b_{n-1}, b_0, b_1, \ldots, b_{i-1})$$

has digit sum $r + 1$ and so is likewise in $S_k^{r,r+1}(n-1)$. Finally, there is a directed path from

$$(b_0, b_1, \ldots, b_{i-1}, b_i + 1, b_{i+1} - 1, b_{i+2}, \ldots, b_{n-1}),$$

to this latter n-tuple via cyclic shifting and so the desired directed path exists. This completes the proof.

□

We can now give the following result.

Theorem 4.5. *Suppose $n \geq 2$ and $k > 2$. Then:*

(i) $S_k^{r,r}(n-1)$ *is Eulerian if and only if* $r \in \{0, 1, (k-1)n-1, (k-1)n\}$*;*

(ii) $S_k^{r,r+1}(n-1)$ *is Eulerian for every* r *(*$0 \leq r < (k-1)n$*).*

Proof. (i) If $r = 0$ or $r = (k-1)n$ then the only n-tuple with digit sum r is 0^n or $(k-1)^n$, respectively. The result follows. Somewhat similarly, the only n-tuple with digit sum $r = 1$ consists of $n-1$ zeros and a single 1 symbol. Clearly this set of n n-tuples can be arranged in a single cycle — a necklace. A precisely analogous argument holds for $r = (k-1)n - 1$.

Now suppose $1 < r < n-1$, and hence $n \geq 4$. By Lemma 4.3, the only cycles in $S_k^{r,r}(n-1)$ consist of necklaces, and thus have period dividing n. However, if $1 < r < (k-1)n - 1$ then $S_k^{r,r}(n-1)$ will clearly contain more than n edges (n-tuples), and hence it contains more than one necklace. However, two necklaces in $S_k^{r,r}(n-1)$ cannot share a vertex since, because w_s is one-to-one, there is only at most one edge of weight r outgoing from (and incoming to) any vertex. The desired result follows.

(ii) Just as previously, because of Lemma 4.2, establishing the Eulerian property just requires us to show that the subgraphs are strongly connected. Suppose $\mathbf{a}$ and $\mathbf{b}$ are vertices in $S_k^{r,r+1}(n-1)$. Then, by Lemma 4.4(i) and (ii), there are directed paths in $S_k^{r,r+1}(n-1)$ from $\mathbf{a}$ to $\mathbf{a}'$ and from $\mathbf{b}'$ to $\mathbf{b}$, where $\mathbf{a}'$ has digit sum r and $\mathbf{b}'$ has digit sum $r+1$. So to establish the result we simply need to show there is a path in $S_k^{r,r+1}(n-1)$ from any n-tuple with digit sum r to any n-tuple with digit sum $r+1$.

Suppose $\mathbf{a}$ has digit sum r and $\mathbf{b}$ has digit sum $r+1$, where $r = s(k-1)+t$, $s \geq 0$, and $0 \leq t < k-1$. Then we claim there are directed paths in $S_k^{r,r+1}(n-1)$ from $\mathbf{a}$ to $((k-1)^s, t, 0^{n-s-1})$, and from $((k-1)^s, t+1, 0^{n-s-1})$ to $\mathbf{b}$. The first assertion follows from repeated application of Lemma 4.4(iii) to $\mathbf{a}$ until an n-tuple $(c_0, c_1, \ldots, c_{n-1})$ of digit sum r is reached for which there is no i such that $c_i < k-1$ and $c_{i+1} > 0$. It follows that $\mathbf{c} = ((k-1)^s, t, 0^{n-s-1})$, and the first claim is established. An exactly analogous argument using repeated applications of Lemma 4.4(iv) shows that there is a directed path from $((k-1)^s, t+1, 0^{n-s-1})$ to $\mathbf{b}$. That is, to complete the proof it remains to show that there is a directed path in $S_k^{r,r+1}(n-1)$ from $((k-1)^s, t, 0^{n-s-1})$ to $((k-1)^s, t+1, 0^{n-s-1})$; but this is immediate since they differ in only one digit, and hence the result follows from Lemma 2.1.

□

Clearly Result 4.1 follows from Theorem 4.5 as a special case.

Analogously to the Hamming weight case, we have the following simple corollary, whose proof is immediate using the same arguments as for Corollary 3.5.

Corollary 4.6. *Suppose $n \geq 2$, $k > 2$ and $0 \leq r < s \leq (k-1)n$. Then $S_k^{r,s}(n-1)$ is Eulerian.*

5 Partitioning using pseudoweight

The third 'measure' for n-tuples that we consider is the pseudoweight, as in Definition 2.3. In this case we need the following.

Definition 5.1 (The pseudoweight subgraph)**.** Suppose $k > 2$, $n \geq 2$ and $0 \leq r \leq s \leq (k-1)n$ (where $r \in \mathbb{Z}$ if k is even and $r = t/2$ for some $t \in \mathbb{Z}$ if k is odd). We write $P_k^{r,s}(n-1)$ for the subgraph of $B_k(n-1)$ consisting of the edges corresponding to n-tuples $\mathbf{a}$ satisfying $r \leq w_p(\mathbf{a}) \leq s$.

Exactly as for digit sum, a special case of the problem we address here has been previously considered for this function. The discussion following [16, Theorem 3.8] establishes the following result.

Result 5.1 ([16])**.** *Suppose $k \geq 2$ and $n \geq 2$. Then $P_k^{n,s}(n-1)$ is Eulerian, where*

$$s = \begin{cases} \frac{nk-1}{2} & \text{if } k \text{ is odd} \\ \frac{nk-2}{2} & \text{if } k \text{ is even} \end{cases}$$

Remark 5.1. Since $\frac{nk}{2}$ is the average value of the function w_p when calculated over all n-tuples, this simply says that the subgraph of $B_k(n-1)$ consisting of all n-tuples with pseudoweight less than the average value is Eulerian.

If $k = 2$ then $f(0) = f(1) = 1$ and the pseudoweight function is trivial and uninteresting. Thus, in the remainder of this discussion we restrict our attention to $k > 2$.

First, analogously to Lemmas 3.1 and 3.2 we have the following.

Lemma 5.2. *If $k > 2$, $n \geq 2$ and $n \leq r \leq s \leq n(k-1)$ (where $r, s \in \mathbb{Z}$ if k is even and $r = t/2$ and $s = u/2$ for some $t, u \in \mathbb{Z}$ if k is odd) then $P_k^{r,s}(n-1)$ is balanced.*

Proof. The fact that $P_k^{r,r}(n-1)$ is balanced for any r is immediate since if $(a_1, a_2, \ldots, a_{n-1})$ is a vertex in $B_k(n-1)$ and $x \in \mathbb{Z}_k$, then $(x, a_1, a_2, \ldots, a_{n-1})$ is an incoming edge in $P_k^{r,r}(n-1)$ if and only if $(a_1, a_2, \ldots, a_{n-1}, x)$ is an outgoing edge in $P_k^{r,r}(n-1)$, since they have the same pseudoweight. But the set of edges in $P_k^{r,s}(n-1)$ is simply the union of the sets of edges in $P_k^{t,t}(n-1)$ for every t satisfying $r \leq t \leq s$, and the result follows. $\square$

Lemma 5.3. *Suppose $k > 2$, $n \geq 2$, $n \leq r \leq n(k-1)$ and $(a_1, a_2, \ldots, a_{n-1})$ is a vertex in $P_k^{r,r}(n-1)$. If $(a_0, a_1, \ldots, a_{n-1})$ is an incoming edge to this vertex then $(a_1, a_2, \ldots, a_n)$ is an outgoing edge if and only if:*

$$a_n = \begin{cases} 0 \text{ or } k/2 & \text{if } k \text{ is even and } a_0 = 0 \text{ or } a_0 = k/2 \\ a_0 & \text{otherwise} \end{cases}$$

Proof. Since both $(a_0, a_1, \ldots, a_{n-1})$ and $(a_1, a_2, \ldots, a_n)$ have the same pseudoweight, it follows that $f(a_0) = f(a_n)$, where f is as defined in Definition 2.3. Hence, unless k is even and $f(a_0) = f(a_n) = k/2$, this automatically implies $a_0 = a_n$. Conversely, if $f(a_0) = f(a_n)$ then $(a_0, a_1, \ldots, a_{n-1})$ and $(a_1, a_2, \ldots, a_n)$ have the same pseudoweight, and the result follows. $\square$

The analysis for this function is somewhat more complex. However, we do have a result similar to Theorems 3.4(i) and 4.5.

Theorem 5.4. *Suppose $n \geq 2$, $k > 2$, and $n \leq r \leq n(k-1)$ (where $r \in \mathbb{Z}$ if k is even and $r = t/2$ for some $t \in \mathbb{Z}$ if k is odd). Then $P_k^{r,r}(n-1)$ is Eulerian if and only if*

(i) $r \in \{n, (k-1)n\}$,

(ii) $r \in \{n+1, (k-1)n-1\}$ *and* $k \neq 3$, *or*

(iii) $r \in \{n-1+k/2, (k-1)(n-1)+k/2\}$ *and* k *odd.*

Proof. Because of Lemma 5.2, to show the Eulerian property holds it suffices to show that the relevant subgraph is strongly connected.

(i) $P_k^{n,n}(n-1)$ consists of a single edge, namely 1^n — a loop from the vertex 1^{n-1} to itself; this is clearly Eulerian. Similarly, the subgraph $P_k^{(k-1)n,(k-1)n}(n-1)$ consists of the single edge $(k-1)^n$ and thus is also Eulerian.

(ii) $P_4^{n+1,n+1}(n-1)$ contains all the n-tuples in the two necklaces $[1^{n-1}, 2]$ and $[1^{n-1}, 0]$. These two necklaces share a vertex, namely 1^{n-1}, and hence the subgraph is strongly connected. If $k > 4$, then $P_k^{n+1,n+1}(n-1)$ contains only a single necklace, namely $[1^{n-1}, 2]$, and thus the subgraph is strongly connected. Analogous arguments hold for $P_4^{3n-1,3n-1}(n-$

1), which contains the necklaces $[3^{n-1}, 2]$ and $[3^{n-1}, 0]$, and $P_k^{n(k-1)-1,n(k-1)-1}(n-1)$, which contains the single necklace $[(k-1)^{n-1}, k-2]$.

(iii) If k is odd then $n-1+k/2$ is not an integer, and hence the only necklace in $P_k^{n-1+k/2,n-1+k/2}(n-1)$ is $[1^{n-1}, 0]$, and so $P_k^{n-1+k/2,n-1+k/2}(n-1)$ is Eulerian. Similarly, $(k-1)(n-1)+k/2$ is not an integer, and hence the only necklace in $P_k^{(k-1)(n-1)+k/2,(k-1)(n-1)+k/2}(n-1)$ is $[(k-1)^{n-1}, 0]$ and so $P_k^{(k-1)(n-1)+k/2,(k-1)(n-1)+k/2}(n-1)$ is Eulerian.

It remains to show that in all other cases $P_k^{r,r}(n-1)$ is not Eulerian. First consider the special case $k = 3$ and $r \in \{n+1, (k-1)n-1\}$. Observe that $P_3^{n+1,n+1}$ contains two necklaces: $[1^{n-1}, 2]$ and $[1^{n-2}, 0, 0]$. These clearly do not share a vertex, and hence $P_3^{n+1,n+1}$ is not Eulerian. Similarly, $P_3^{2n-1,2n-1}(n-1)$ contains the two necklaces. $[2^{n-1}, 1]$ and $[2^{n-2}, 0, 0]$. These clearly do not share a vertex, and hence $P_3^{2n-1,2n-1}(n-1)$ is not Eulerian.

Next, suppose k is odd, $n+1 < r < (k-1)n-1$ and $r \notin \{n-1+k/2, (k-1)(n-1)+k/2\}$. In this case $P_k^{r,r}(n-1)$ will always contain more than one necklace. However, no two necklaces can share a vertex since w_p is one-to-one when k is odd, and hence a vertex can only have at most one incoming or outgoing edge with any given pseudoweight. Hence, $P_k^{r,r}(n-1)$ is not strongly connected.

Finally, suppose k is even and $n+1 < r < (k-1)n-1$. Suppose $r = n+s(k-2)+t-1$, where $1 \leq t < k-1$ and $0 \leq s < n$. Then there exists an edge in $P_k^{r,r}(n-1)$ of the form $((k-1)^s, t, 1^{n-s-1})$. No vertex in this necklace will have more than one incoming edge or outgoing edge in $P_k^{r,r}(n-1)$ except in the case $t = k/2$. If $t = k/2$ then the necklace $[(k-1)^s, 0, 1^{n-s-1}]$ will share a vertex with the necklace $[(k-1)^s, t, 1^{n-s-1}]$ but neither necklace will share a vertex with any other edges in $P_k^{r,r}(n-1)$.

We next show that $P_k^{r,r}(n-1)$ will always contain another necklace not equal to either $[(k-1)^s, t, 1^{n-s-1}]$ (or $[(k-1)^s, 0, 1^{n-s-1}]$ if $t = k/2$), establishing that $P_k^{r,r}(n-1)$ is not strongly connected. We need to consider three cases.

- First suppose $s = 0$, i.e. there exists an edge in $P_k^{r,r}(n-1)$ of the form $(t, 1^{n-1})$, where $t > 2$ since $r > n+1$. Then there will also exist an edge of the form $(t-1, 2, 1^{n-2})$ in $P_k^{r,r}(n-1)$, since $n \geq 2$, i.e. another necklace.

- Next suppose $s > 0$ and $t < k-2$; then, since $k \geq 4$, there exists a further edge of the form $((k-1)^{s-1}, k-2, t+1, 1^{n-s-1})$ in $P_k^{r,r}(n-1)$.

- Finally suppose $s > 0$ and $t = k-2$. Now, since $r < (k-1)n-1$, $n-s-1 > 0$. Then the edge mentioned above takes the form $((k-1)^s, k-2, 1^{n-s-1})$. An additional edge is then $((k-1)^s, k-3, 2, 1^{n-s-2})$.

The result follows. $\square$

The k even case is somewhat simpler than the k odd case, since if k is even then $w_p(\mathbf{a}) \in \mathbb{Z}$ for any k-ary n-tuple $\mathbf{a}$. We first need the following preliminary lemma.

Lemma 5.5. *Suppose $n \geq 2$, $k > 2$ is even, and $n \leq r \leq n(k-1)$. Suppose $\mathbf{a}$ is an edge (i.e. a k-ary n-tuple) in $P_k^{r,r}(n-1)$ and $\bar{\mathbf{a}}$ is equal to $\mathbf{a}$ with every 0 changed to $k/2$. Then there exist directed paths in $P_k^{r,r}(n-1)$ from $\mathbf{a}$ to $\bar{\mathbf{a}}$ and from $\bar{\mathbf{a}}$ to $\mathbf{a}$.*

Proof. The result follows using repeated applications of Lemma 2.1. That is, to obtain a directed path from $\mathbf{a}$ to $\bar{\mathbf{a}}$, simply use Lemma 2.1 to obtain a directed path from $\mathbf{a}$ to an edge $\mathbf{a}'$ equal to $\mathbf{a}$ with a single 0 changed to $k/2$, and then a directed path from $\mathbf{a}'$ to an edge $\mathbf{a}''$ equal to $\mathbf{a}'$ with another 0 changed to $k/2$, and so on until $\bar{\mathbf{a}}$ is reached. For the path in the opposite direction simply change the relevant entries of $k/2$ to 0 one by one until $\mathbf{a}$ is reached. $\square$

We can now give the main result.

Theorem 5.6. *Suppose $n \geq 2$ and $k > 2$ is even. Then $P_k^{r,r+1}(n-1)$ is Eulerian for every $r \in \mathbb{Z}$ satisfying $n \leq r < (k-1)n$.*

Proof. By Lemma 5.2 we need only show that $P_k^{r,r+1}(n-1)$ is strongly connected. Also, by Lemma 5.5, we need only show that there is a directed path between any two edges in $P_k^{r,r+1}(n-1)$ whose corresponding n-tuples do not contain any zeros. But there is a simple graph isomorphism between the subgraph of $P_k^{r,r+1}(n-1)$ whose edges are the n-tuples containing no zeros and $S_{k-1}^{r-n,r-n+1}(n-1)$ (by subtracting one from every entry in an n-tuple), and so the result follows from Theorem 4.5(ii). $\square$

To analyse the k odd case we first need the following.

Lemma 5.7. *Suppose $n \geq 2$, $k > 2$ is odd, and $n + (k-1)/2 \leq r \leq n(k-1) - (k-2)/2$, where $r = t/2$ for some $t \in \mathbb{Z}$. Then:*

(i) if $\mathbf{a}$ has pseudoweight r, there exists a directed path in $P_k^{r,r+0.5}(n-1)$ from $\mathbf{a}$ to $\mathbf{a}'$, where $\mathbf{a}'$ contains no entries equal to 0;

(ii) if $\mathbf{b}$ has pseudoweight $r+0.5$, there exists a directed path in $P_k^{r,r+0.5}(n-1)$ from $\mathbf{b}'$ to $\mathbf{b}$, where $\mathbf{b}'$ contains no entries equal to 0.

Proof. (i) First observe that, for any k-ary n-tuple $\mathbf{a}$ of pseudoweight s, replacing a single zero in $\mathbf{a}$ by $(k-1)/2$ results in in a k-ary n-tuple of pseudoweight $s - 0.5$, and replacing a single zero by $(k+1)/2$ results in in a k-ary n-tuple of pseudoweight $s + 0.5$. So, starting from $\mathbf{a}$ (of pseudoweight r), replace every 0 one by one with alternately $(k+1)/2$ and $(k-1)/2$ until an n-tuple is reached (which we label $\mathbf{a}'$) containing no 0 entries. In every case the pseudoweight of the result will be either r or $r + 0.5$, and hence by Lemma 2.1 there is a directed path from the starting n-tuple to the modified n-tuple in $P_k^{r,r+0.5}(n-1)$. Joining these paths together yields the desired directed path from $\mathbf{a}$ to $\mathbf{a}'$.

(ii) The argument for (i) works in reverse, replacing every 0 entry in $\mathbf{b}$ with alternately $(k-1)/2$ and $(k+1)/2$, yielding a directed path in $P_k^{r,r+0.5}(n-1)$ starting at an n-tuple $\mathbf{b}'$ containing no entries equal to 0 and ending in $\mathbf{b}$.

□

We can now give a companion result to Theorem 5.6.

Theorem 5.8. *Suppose $n \geq 2$ and $k > 2$ is odd. Then $P_k^{r,r+1}(n-1)$ is Eulerian for every $r \in \mathbb{Z}$ satisfying $n \leq r < (k-1)n$.*

Proof. As previously, by Lemma 5.2 we need only show that $P_k^{r,r+1}(n-1)$ is strongly connected. Suppose $\mathbf{a}$ and $\mathbf{b}$ are distinct edges in $P_k^{r,r+1}(n-1)$.

If $\mathbf{a}$ has no zeros then set $\mathbf{a}^* = \mathbf{a}$. Otherwise, if $\mathbf{a}$ has pseudoweight r or $r+0.5$, then, by Lemma 5.7, there is a directed path in $P_k^{r,r+1}(n-1)$ from $\mathbf{a}$ to an edge $\mathbf{a}^*$ which is zero-free. Alternatively, if $\mathbf{a}$ has pseudoweight $r+1$ then let $\mathbf{a}'$ equal $\mathbf{a}$ with one zero replaced by $(k-1)/2$. By Lemma 2.1, since $\mathbf{a}'$ has pseudoweight $r+0.5$, there is a directed path in $P_k^{r,r+1}(n-1)$ from $\mathbf{a}$ to $\mathbf{a}'$. Moreover, by Lemma 5.7 there is a directed path in $P_k^{r,r+1}(n-1)$ from $\mathbf{a}'$ to an edge $\mathbf{a}^*$ which is zero-free. That is, in every case there is a directed path in $P_k^{r,r+1}(n-1)$ from $\mathbf{a}$ to an edge $\mathbf{a}^*$ which is zero-free.

By a precisely analogous argument there is a directed path in $P_k^{r,r+1}(n-1)$ from an edge $\mathbf{b}^*$ containing no zeros to $\mathbf{b}$.

But there exists a directed path in $P_k^{r,r+1}(n-1)$ from $\mathbf{a}^*$ to $\mathbf{b}^*$ using exactly the same argument as employed in Theorem 5.6 (observing that r is an integer). The result follows. □

The separate results for k odd and k even (Theorems 5.6 and 5.8) can be combined to give the following, whose proof is immediate using the same arguments as for Corollary 3.5.

Corollary 5.9. *Suppose $n \geq 2$, $k > 2$ and $n \leq r < s \leq (k-1)n$, where $r, s \in \mathbb{Z}$. Then $P_k^{r,s}(n-1)$ is Eulerian.*

6 Concluding remark

It would be of interest to consider whether similar results could be obtained for other possible permutation-agnostic weight functions.

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