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Weight-constrained universal sequences

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Abstract

In 2011/12, Ruskey, Sawada, Stevens and Williams showed that a binary universal sequence can be constructed containing all n-tuples of Hamming weight either r or $r+1$ for every possible r, and no others. In this paper we examine extensions of this result that require choosing a weight-like function applying to n-tuples with symbols taken from an alphabet of arbitrary (finite) size. We consider three possible such weight functions, and in each case establish an analogous result to that of Ruskey, Sawada, Stevens and Williams.

1 Introduction

The main focus of this paper is a special class of *universal sequences* (or *universal cycles*), namely — at least for the purposes of this paper — infinite periodic sequences with elements from a finite set in which every n-tuple of a certain type occurs exactly once. Universal sequences of a range of types have been widely studied — see, for example, Chung et al. [8]. According to Hollmann and van Lint [13], the term universal sequence in this context dates back to a 1985 paper of Lempel and Cohn [16], although they use the term in a much more restricted sense. The more general definition we use here is consistent with the rather abstract definition given by Chroman et al. [7].

In this paper we are interested in sequences including all n-tuples for which some weight-like measure is restricted. Such sequences have been previously considered in the binary case by Sawada et al. [26] and Ruskey et al. [24], who considered binary n-tuples of restricted Hamming weight. In this paper we generalise the main existence result of Sawada et al. and Ruskey et al. to finite alphabets of arbitrary size. In doing so we look at three main examples of weight-like functions, one case of which was previously considered by Gabrić et al. [11]. Results of this type can also be derived from the work of

Sawada et al. [28] developed in a slightly different context, as we discuss in Section 1.3 below.

In particular we show that, for all three weight-functions, and for all integer pairs r, s $(r < s)$ where r and s are possible weights for the relevant weight function, it is possible to construct a universal sequence containing all n-tuples with weights in the range $[r, s]$. Precise versions of these results can be found in Corollaries 3.5, 4.6, and 5.9.

1.1 Background

In this paper we consider periodic sequences (s_i) with elements from $\mathbb{Z}_k$ for some k, which we refer to as k-ary. For a sequence $S = (s_i)$ and $n \geq 1$ we write $\mathbf{s}_n(i) = (s_i, s_{i+1}, \ldots, s_{i+n-1})$, for an n-tuple corresponding to a substring or factor of n consecutive symbols occurring in the sequence at positions $i, i+1, \ldots, i+n-1$. We say such an n-tuple appears in the sequence. Observe that in the remainder of the paper we write x^m as a shorthand for m consecutive repetitions of the symbol x.

Definition 1.1 ([2])**.** A k-ary *n-window sequence* $S = (s_i)$ is a periodic sequence of elements from $\mathbb{Z}_k$ $(k > 1,\ n > 1)$ with the property that no n-tuple appears more than once in a period of the sequence, i.e. with the property that if $\mathbf{s}_n(i) = \mathbf{s}_n(j)$ for some i, j, then $i \equiv j \pmod{m}$ where m is the period of the sequence. Such a sequence is also known as a *cut-down de Bruijn sequence* (see, for example, Cameron et al. [5]), and this is the term we use here.

A k-ary de Bruijn sequence *of span n* is then simply an n-window sequence in which every k-ary n-tuple appears once in a period, i.e. an n-window sequence of maximal period.

Following Alhakim et al. [2] and many previous authors, we also introduce the k-ary de Bruijn digraph. For positive integers $n \geq 1$ and $k \geq 2$, let $\mathbb{Z}_k^n$ be the set of all k^n tuples of length n with entries from the group $\mathbb{Z}_k$ of residues modulo k. The order n de Bruijn digraph, $B_k(n)$, is a directed graph with $\mathbb{Z}_k^n$ as its vertex set in which, for any two vertices $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$ and $\mathbf{y} = (y_0, y_1, \ldots, y_{n-1})$, the pair $(\mathbf{x}, \mathbf{y})$ is an edge if and only if $y_i = x_{i+1}$ for every i $(0 \leq i < n-1)$. We label such an edge with the $(n+1)$-tuple $(x_0, x_1, \ldots, x_{n-1}, y_{n-1})$.

Note that we have defined two ways of specifying an edge in $B_k(n)$, namely as either a pair of vertices $(\mathbf{a}, \mathbf{b})$, where $\mathbf{a}, \mathbf{b}$ are k-ary n-tuples, or as a single k-ary $(n+1)$-tuple $\mathbf{x}$. We refer to an edge in $B_k(n)$ and a k-ary $(n+1)$-tuple interchangeably throughout.

Definition 1.2. An *Eulerian digraph* is a strongly connected digraph, i.e. one in which there is a directed path between any pair of vertices, for which

every vertex has in-degree equal to out-degree. This latter property characterises a *balanced* digraph.

The name derives from the fact that there exists an Eulerian circuit, i.e. a closed directed path visiting every edge once, in a digraph if and only if the digraph is Eulerian — see, for example, Corollary 6.1 of Gibbons [12] or Knuth [15, Section 2.3.4.2, Theorem G]. Moreover, there are simple and efficient algorithms for finding Eulerian circuits — see for example [12, Figure 6.5] or [15, Section 2.3.4.2, Theorem D].

There is then a simple correspondence between a directed Eulerian circuit in $B_k(n-1)$ and a de Bruijn sequence of span n; this generalises in an obvious way so that a directed circuit (or cycle) in $B_k(n-1)$ of length $m \leq k^n$ corresponds to a cut-down de Bruijn sequence of period m.

We also need the following.

Definition 1.3 (Necklace — see, for example, [23])**.** A k-ary *necklace* within the de Bruijn graph $B_k(n-1)$, written as $[a_0, a_1, \ldots, a_{m-1}]$, where $m \mid n$, consists of all distinct n-tuples of the form $(a_{\overline{i}}, a_{\overline{i+1}}, \ldots, a_{\overline{i+n-1}})$, for $0 \leq i < m$, i.e. all cyclic shifts, where $\overline{s} = s \bmod m$.

De Bruijn sequences have a range of real-world applications, ranging from position location on a linear track or rotating shaft [3, 4, 20, 22, 31], to genome assembly [9]. As observed by Cameron et al. [5], 'for some applications it may be more convenient to produce a cycle of arbitrary length such that there are no repeated strings', i.e. a cut-down de Bruijn sequence of a particular period that derives from the specifics of the application.

A number of authors have considered the problem of constructing cut-down de Bruijn sequences with particular properties. Jackson et al. [14, Problem 477] observed that a binary n-window sequence exists for every possible period m satisfying $n \leq m \leq 2^n$. Sawada et al. [26] and Ruskey et al. [24] showed that a cut-down de Bruijn sequence can be constructed containing all n-tuples with weights in the range $[a, b]$ as long as $0 \leq a < b \leq n$, and such sequences were further studied by Li et al. [17]. Later, Gabrić et al. [11], generalised this result to alphabets of an arbitrary size in the case where weight is defined as the digit sum, and showed that all n-tuples bounded above (or below) by a given weight can be arranged in a cut-down de Bruijn sequence. Even more recently, Gabrić et al. [10] showed how to construct certain classes of universal sequences that can be readily decoded.

Sequences containing only n-tuples with specified weights are a particular example of what are sometimes known as 'de Bruijn sequences with forbidden subsequences', as for example examined in the binary case by Penne [21] and for arbitrary alphabets by Alhakim [1]. Tan and Shallit [30], studied the more general problem of the existence of sequences containing only the

members of a pre-specified set of n-tuples, where they allow such n-tuples to occur multiple times in a period. These are all examples of universal sequences, as introduced at the beginning of this paper.

1.2 Cut down de Bruijn sequences in weight-restricted sub-graphs

As observed at the start of this paper, we are concerned with a particular class of cut-down de Bruijn sequences, as previously considered by Sawada et al. [26] and Ruskey et al. [24]. They showed that, if $n \geq 1$ and $0 \leq r < n$, the sub-graph of the de Bruijn graph $B_2(n-1)$ consisting of all edges corresponding to n-tuples of Hamming weights r and $r+1$ is Eulerian, and hence there exists a cut down de Bruijn sequence which includes all n-tuples of weights r and $r+1$ (and no others).

In this paper we consider to what extent a similar result holds for the k-ary de Bruijn graph. Of course, in this case there are a number of ways of computing the 'weight' of an n-tuple, and we therefore consider what happens for three different functions that assign a weight-like value to an n-tuple.

1.3 The Sawada-Williams-Wong result

Sawada et al. [28] considered sets S of k-ary n-tuples satisfying two closure properties:

- S is closed under cyclic rotation of n-tuples;
- S is a $(k-1)$-suffix language, i.e. if $(a_0, a_1, \ldots, a_{n-1}) \in S$ then

$$(a_0, a_1, \ldots, a_{n-i-1}, (k-1)^i) \in S$$

  for every $i \geq 0$, that is S is closed under replacement of the last i entries of any n-tuple by a string of i values equal to $k-1$.

They refer to the class of all such sets S as $\mathcal{C}(n,k)$. Using slightly different language they show the following.

Result 1.1 (Corollary 3 of [28])**.** *If $S \in \mathcal{C}(n,k)$ and H is the subgraph of $B_k(n-1)$ whose edges correspond to the n-tuples in S, then H is Eulerian, and hence there exists a universal cycle containing precisely the n-tuples in S.*

In the discussions below we indicate how this relates to results for all three of the weight functions we consider here.

2 Weight functions for k-ary alphabets

There are many ways of generalising the notion of Hamming weight of a binary n-tuple to the k-ary case. We briefly introduce some key possibilities here. The 'standard' generalisation of Hamming weight to the k-ary case is as follows.

Definition 2.1 (Hamming weight)**.** Suppose $k \geq 2$ and $n \geq 1$. If $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ is a k-ary n-tuple, then the *Hamming weight* $w_h(\mathbf{a})$ of $\mathbf{a}$ is equal to the number of values i for which $a_i \neq 0$.

It should be clear that in the binary case, i.e. where $k = 2$, the Hamming weight corresponds to the number of ones in the n-tuple, i.e. the expected definition. An alternative measure for n-tuples, which also corresponds to Hamming weight in the case $k = 2$, is as follows.

Definition 2.2 (Digit sum)**.** Suppose $k \geq 2$, $n \geq 1$ and $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ is a k-ary n-tuple. Then the *digit sum* (or *symbol sum*) of $\mathbf{a}$ is defined as

$$w_s(\mathbf{a}) = \sum_{i=0}^{n-1} a_i$$

where we compute the sum in $\mathbb{Z}$ and treat the values a_i as integers in the range $[0, k-1]$.

Remark 2.1. Note that the digit sum is referred to simply as the *weight* of an n-tuple in [18, Definition 11]. It should be clear that the digit sum is also a natural generalisation of Hamming weight for the binary case.

Yet another alternative way of assigning a numeric value to an n-tuple is as follows.

Definition 2.3 ([18], Definition 10)**.** Suppose $k \geq 2$, $n \geq 1$ and $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ is a k-ary n-tuple. Define the function $f : \mathbb{Z}_k \to \mathbb{Q}$ as follows: for any $a \in \mathbb{Z}_k$ treat a as an integer in the range $[0, k-1]$ and set $f(a) = a$ if $a \neq 0$ and $f(a) = k/2$ if $a = 0$. Then the *pseudoweight* of $\mathbf{a}$ is defined to be the sum

$$w_p(\mathbf{a}) = \sum_{i=0}^{n-1} f(a_i)$$

where the sum is computed in $\mathbb{Q}$.

This apparently curious notion was introduced to help generate special classes of cut-down de Bruijn sequences known as orientable sequences and negative orientable sequences (see, for example, [19]).

In the remainder of this paper we consider to what extent generalised versions of the Sawada et al. [26] and Ruskey et al. [24] result hold for these three weight functions. We first need the following preliminary definition and result that simplify arguments for all three weight functions we examine.

Definition 2.4. Suppose $k \geq 2$, $n \geq 1$ and w is function mapping k-ary n-tuples to $\mathbb{Q}$. Then w is said to be *permutation-agnostic* if, for any k-ary n-tuple $\mathbf{a}$, $w(\mathbf{a}) = w(p(\mathbf{a}))$ for every permutation p acting on the n entries in $\mathbf{a}$.

The three weight functions w_h, w_s and w_p are all permutation-agnostic since they are all defined as a function of the elements of an n-tuple in a way that does not depend on the ordering. We observe the following result applying to all such weight functions.

Lemma 2.1. *Suppose $k \geq 2$, $n \geq 2$, $r, s \in \mathbb{Q}$ ($r \leq s$), and w is permutation-agnostic function mapping k-ary n-tuples to $\mathbb{Q}$. Let $W_k^{r,s}(n-1)$ be the subgraph of $B_k(n-1)$ consisting of the edges corresponding to n-tuples $\mathbf{a}$ satisfying $r \leq w(\mathbf{a}) \leq s$. Then:*

(i) if $\mathbf{a}$ and $\mathbf{b}$ are edges (n-tuples) in $W_k^{r,s}(n-1)$ that satisfy $w(\mathbf{a}) = r$ and $w(\mathbf{b}) = s$ and that differ in a single position, then there are directed paths from $\mathbf{a}$ to $\mathbf{b}$ and from $\mathbf{b}$ to $\mathbf{a}$ in $W_k^{r,s}(n-1)$;

(ii) $W_k^{r,s}(n-1)$ is balanced.

Proof.

- Suppose $\mathbf{a} = (a_0, a_1 \ldots, a_{n-1})$ and $\mathbf{b} = (b_0, b_1 \ldots, b_{n-1})$, where $a_i = b_i$ for every i except $i = t$ for some t ($0 \leq t \leq n-1$). Since w is permutation-agnostic, if $\mathbf{a}'$ is a cyclic shift of $\mathbf{a}$ then $w(\mathbf{a}) = w(\mathbf{a}')$. Hence there is a directed path from $\mathbf{a}$ to $(a_t, a_{t+1}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{t-1})$ in $W_k^{r,s}(n-1)$. Clearly $(a_t, a_{t+1}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{t-1})$ is an incoming edge to the vertex $(a_{t+1}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{t-1})$ and $(a_{t+1}, a_{t+2}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{t-1},$ $(b_{t+1}, b_{t+2}, \ldots, b_{n-1}, b_0, b_1, \ldots, b_t)$ is an outgoing edge from this vertex. That is, we can extend the directed path so that it goes from $\mathbf{a}$ to a cyclic shift of $\mathbf{b}$. Using the same argument as previously, there is a directed path from $(b_{t+1}, b_{t+2}, \ldots, b_{n-1}, b_0, b_1, \ldots, b_t)$ to $\mathbf{b}$ in $W_k^{r,s}(n-1)$. The same argument works in reverse, and the result follows.

- $W_k^{r,r}(n-1)$ is balanced for any r, since if $(a_1, a_2, \ldots, a_{n-1})$ is a vertex in $B_k(n-1)$ and $x \in \mathbb{Z}_k$, then $(x, a_1, a_2, \ldots, a_{n-1})$ is an incoming edge in $W_k^{r,r}(n-1)$ if and only if $(a_1, a_2, \ldots, a_{n-1}, x)$ is an outgoing edge in $W_k^{r,r}(n-1)$ since w is permutation-agnostic. But the set of edges in $W_k^{r,s}(n-1)$ is simply the disjoint union of the sets of edges in $W_k^{t,t}(n-1)$ for every t ($r \leq t \leq s$), and the result follows.

□

3 Partitioning using Hamming weight

We start by considering the case where the weight of an n-tuple is simply the number of non-zero entries, i.e. using the function w_h as specified in Definition 2.1. We have a result very similar to that of Sawada et al. [26] and Ruskey et al. [24] for the case $k = 2$. We first need the following notation.

Definition 3.1 (The Hamming weight subgraph)**.** Suppose $k \geq 2$, $n \geq 1$ and $0 \leq r \leq s \leq n$. We write $H_k^{r,s}(n-1)$ for the subgraph of $B_k(n-1)$ consisting of the edges corresponding to n-tuples $\mathbf{a}$ satisfying $r \leq w_h(\mathbf{a}) \leq s$.

Remark 3.1. $H_2^{r-1,r}(n-1)$ is what Ruskey et al. [24] refer to as $G(\mathbf{B}_{r-1}^r(n-1))$.

The following two results are straightforward to establish.

Lemma 3.1. *Suppose $k \geq 2$, $n \geq 1$ and $0 \leq r \leq s \leq n$. Then the number of edges in $H_k^{r,s}(n-1)$ is*

$$\sum_{i=r}^{s} \binom{n}{i} (k-1)^i.$$

Proof. Observe that $H_k^{r,r}(n-1)$ contains

$$\binom{n}{r} (k-1)^r$$

edges, since the number of such edges is simply the number of ways of choosing r non-zero elements from n ($\binom{n}{r}$) multiplied by the number of choices for these non-zero elements ($(k-1)^r$). But the edges in $H_k^{r,s}(n-1)$ are simply all those in $H_k^{i,i}(n-1)$ for every i in the range $[r, s]$, and the result follows. □

Lemma 3.2. *Suppose $k \geq 2$, $n \geq 2$, $0 \leq r \leq n$ and $(a_1, a_2, \ldots, a_{n-1})$ is a vertex in $H_k^{r,r}(n-1)$. If $(a_0, a_1, \ldots, a_{n-1})$ is an incoming edge to this vertex then $(a_1, a_2, \ldots, a_n)$ is an outgoing edge if and only if either $a_0 = a_n = 0$ or a_0 and a_n are both non-zero.*

Proof. This follows since all edges in $H_k^{r,r}(n-1)$ must correspond to n-tuples with precisely r non-zero entries. □

Before giving the main result we need the following key lemma.

Lemma 3.3. *Suppose $n \geq 2$, $k \geq 2$ and $0 \leq r < n$.*

(i) *If* $\mathbf{a}$ *is an edge (i.e. an* n*-tuple) in* $H_k^{r,r+1}(n-1)$*, then there is a directed path in* $H_k^{r,r+1}(n-1)$ *from* $\mathbf{a}$ *to an edge* $\mathbf{a}'$ *satisfying* $w_h(\mathbf{a}') = r$*.*

(ii) *If* $\mathbf{b}$ *is an edge (i.e. an* n*-tuple) in* $H_k^{r,r+1}(n-1)$*, then there is a directed path in* $H_k^{r,r+1}(n-1)$ *from an edge* $\mathbf{b}'$ *satisfying* $w_h(\mathbf{b}') = r+1$ *to* $\mathbf{b}$*.*

(iii) *If* $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ *is an edge (i.e. an* n*-tuple) in* $H_k^{r,r+1}(n-1)$ *such that* $w_h(\mathbf{a}) = r$ *and* $a_i = 0$ *for some* i*, then there is a directed path in* $H_k^{r,r+1}(n-1)$ *from* $\mathbf{a}$ *to*

$$(a_0, a_1, \ldots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \ldots, a_{n-1}),$$

i.e. $\mathbf{a}$ *modified so that* a_i *and* a_{i+1} *are interchanged.*

(iv) *If* $\mathbf{b} = (b_0, b_1, \ldots, b_{n-1})$ *is an edge (i.e. an* n*-tuple) in* $H_k^{r,r+1}(n-1)$ *such that* $w_h(\mathbf{b}) = r+1$ *and* $b_i = 0$ *for some* i*, then there is a directed path in* $H_k^{r,r+1}(n-1)$ *from*

$$(b_0, b_1, \ldots, b_{i-1}, b_{i+1}, b_i, b_{i+2}, \ldots, b_{n-1})$$

i.e. $\mathbf{b}$ *modified so that* b_i *and* b_{i+1} *are interchanged, to* $\mathbf{b}$*.*

Proof. (i) If $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ satisfies $w_h(\mathbf{a}) = r+1$, then suppose $a_i \neq 0$ for some i (which must exist since $w_h(\mathbf{a}) > 0$). Now clearly there is a directed path in $H_k^{r,r+1}(n-1)$ from $\mathbf{a}$ to $(a_i, a_{i+1}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{i-1})$, simply by taking successive cyclic shifts. Because $w_h(\mathbf{a}) = r+1$ and $a_i \neq 0$, the edge $(a_{i+1}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{i-1}, 0)$ is an edge in $H_k^{r,r+1}(n-1)$ with weight r, and so a directed path exists in $H_k^{r,r+1}(n-1)$ from $\mathbf{a}$ to an edge with weight r. Alternatively, if $w_h(\mathbf{a}) = r$, then simply set $\mathbf{a} = \mathbf{a}'$. Either way the result holds.

(ii) If $\mathbf{b} = (b_0, b_1, \ldots, b_{n-1})$ satisfies $w_h(\mathbf{b}) = r < n$, then there is at least one value b_i such that $b_i = 0$. As in the previous case, there is a directed path in $H_k^{r,r+1}(n-1)$ from $(b_i, b_{i+1}, \ldots, b_{n-1}, b_0, b_1, \ldots, b_{i-1})$ to $\mathbf{b}$. Because $w_h(\mathbf{b}) = r$, the edge $(1, b_{i+1}, \ldots, b_{n-1}, b_0, b_1, \ldots, b_{i-1})$ is an edge in $H_k^{r,r+1}(n-1)$ with Hamming weight $r+1$, and the desired directed path exists. Alternatively, if $w_h(\mathbf{b}) = r+1$, then simply set $\mathbf{b} = \mathbf{b}'$. Either way the result holds.

(iii) There is clearly a directed path from $\mathbf{a}$ to $(a_i, a_{i+1}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{i-1})$ in $H_k^{r,r+1}(n-1)$, simply by taking successive cyclic shifts. Since $\mathbf{a}$ has weight r and $a_i = 0$, the n-tuple

$$(a_{i+1}, a_{i+2}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{i-1}, a_{i+1})$$

has weight r or $r+1$ and hence is in $H_k^{r,r+1}(n-1)$. Clearly

$$(a_{i+2}, a_{i+3}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{i-1}, a_{i+1}, a_i)$$

has weight the same as $\mathbf{a}$ and hence is also in $H_k^{r,r+1}(n-1)$. Finally, there is a directed path from this latter n-tuple to

$$(a_0, a_1, \ldots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \ldots, a_{n-1}),$$

via cyclic shifting and so the desired directed path exists.

(iv) There is a directed path from $(b_{i+2}, b_{i+3}, \ldots, b_{n-1}, b_0, b_1, \ldots, b_{i+1})$ to $\mathbf{b}$ in $H_k^{r,r+1}(n-1)$, taking cyclic shifts. Because $\mathbf{b}$ has weight $r+1$ and $b_i = 0$, the n-tuple

$$(b_i, b_{i+2}, \ldots, b_{n-1}, b_0, b_1, \ldots, b_i)$$

has weight r or $r+1$ and hence is in $H_k^{r,r+1}(n-1)$. Also

$$(b_{i+1}, b_i, b_{i+2}, \ldots, b_{n-1}, b_0, b_1, \ldots, b_{i-1})$$

has weight the same as $\mathbf{b}$ and so is likewise in $H_k^{r,r+1}(n-1)$. This completes the proof.

□

We can now give our main result.

Theorem 3.4. *Suppose $k \geq 2$ and $n \geq 2$. Then:*

(i) $H_k^{r,r}(n-1)$ is Eulerian if and only if $r \in \{0, 1, n-1, n\}$;

(ii) $H_k^{r,r+1}(n-1)$ is Eulerian for every r $(0 \leq r < n)$.

Remark 3.2. Certain special cases of this result were already known. For the case $k = 2$ this is precisely [24, Theorem 2.4]. Also, using the notation of Section 1.3, the set of edges in $H_k^{r,n}(n-1)$ is in $\mathcal{C}(n,k)$ for every r, $0 \leq r \leq n$, since replacing entries of an n-tuple with $k-1$ can never decrease the Hamming weight. Hence, by Result 1.1, $H_k^{r,n}(n-1)$ is Eulerian.

Proof. Because of Lemma 2.1, establishing the Eulerian property just requires us to show that the subgraphs are strongly connected.

(i) The result is trivially true for $r = 0$, since there is a single directed edge with Hamming weight 0, the loop to and from the vertex 0^{n-1}. If $r = n$, then the edges in $H_k^{n,n}(n-1)$ are all the n-tuples that do not contain a zero. But this subgraph is isomorphic to the full de Bruijn graph $B_{k-1}(n-1)$, and hence it is clearly strongly connected.

Now suppose $r = 1$. The edges of $H_k^{1,1}(n-1)$ correspond to all n-tuples containing a single non-zero entry. Suppose $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ and $\mathbf{b} = (b_0, b_1, \ldots, b_{n-1})$ are two such n-tuples, where $a_u \neq 0$ and $b_v \neq 0$

and all the other a_i and b_i values are zero. Then there is clearly a directed path in $H_k^{1,1}(n-1)$ from $\mathbf{a}$ to the n-tuple $(a_u, 0, 0, \ldots, 0)$ involving taking successive cyclic shifts of $\mathbf{a}$, and similarly there is also a directed path from the n-tuple $(0, 0, \ldots, 0, b_v)$ to $\mathbf{b}$. Also, $(a_u, 0, 0, \ldots, 0)$ is an incoming edge to the vertex 0^{n-1} and $(0, 0, \ldots, 0, b_v)$ is an outgoing edge from the same vertex. Hence there is a directed path from $\mathbf{a}$ to $\mathbf{b}$ and we have established strong connectivity.

Finally, a somewhat similar argument applies when $r = n-1$. Suppose $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ and $\mathbf{b} = (b_0, b_1, \ldots, b_{n-1})$ are two n-tuples corresponding to edges in $H_k^{n-1,n-1}(n-1)$, where $a_u = b_v = 0$ and all the other a_i and b_i values are non-zero. Then there are directed paths in $H_k^{n-1,n-1}(n-1)$:

- from $\mathbf{a}$ to the n-tuple $(a_{u+1}, a_{u+2}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{u-1}, 0)$, just by taking cyclic shifts;
- then from $(a_{u+1}, a_{u+2}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{u-1}, 0)$ to $(0, b_{v+1}, b_{v+2}, \ldots, b_{n-1}, b_0, b_1, \ldots, b_{v-1})$ (from repeated application of Lemma 3.2); and
- finally from $(0, b_{v+1}, b_{v+2}, \ldots, b_{n-1}, b_0, b_1, \ldots, b_{v-1})$ to $\mathbf{b}$, again taking cyclic shifts;

yielding the required directed path from $\mathbf{a}$ to $\mathbf{b}$.

Finally suppose $1 < r < n-1$, and thus $n \geq 4$ — we need to show that $H_k^{r,r}(n-1)$ is not strongly connected. Regardless of r there will exist two edges in $H_k^{r,r}(n-1)$ corresponding to the n-tuples $\mathbf{a} = (1^r, 0^{n-r})$ and $\mathbf{b} = (1, 0, 1^{r-1}, 0^{n-r-1})$, and all the constituent strings are non-empty since $n \geq 4$ and $1 < r < n-1$. We claim that there is no path in $H_k^{r,r}(n-1)$ linking $\mathbf{a}$ to $\mathbf{b}$.

This holds because, from Lemma 3.2, any edge on a path in $H_k^{r,r}(n-1)$ starting from $\mathbf{a}$ must contain r consecutive non-zero entries (working cyclically). Similarly, any edges on a path starting from $\mathbf{b}$ must contain a zero followed by $r-1$ consecutive non-zero entries and then another zero (again working cyclically). That is $\mathbf{b}$ cannot be on a path starting at $\mathbf{a}$, and vice versa.

(ii) Suppose $\mathbf{a}$ and $\mathbf{b}$ are edges in $H_k^{r,r+1}(n-1)$. Then, by Lemma 3.3(i) and (ii), there are directed paths in $H_k^{r,r+1}(n-1)$ from $\mathbf{a}$ to $\mathbf{a}'$ and from $\mathbf{b}'$ to $\mathbf{b}$, where $\mathbf{a}'$ has weight r and $\mathbf{b}'$ has weight $r+1$. So to establish the result we simply need to show there is a path in $H_k^{r,r+1}(n-1)$ from any n-tuple with weight r to any n-tuple with weight $r+1$.

Suppose $\mathbf{a}$ has weight r and $\mathbf{b}$ has weight $r+1$, where $0 \leq r < n$. Then we claim there are directed paths in $H_k^{r,r+1}(n-1)$ from $\mathbf{a}$ to $(1^r, 0^{n-r})$, and from $(1^{r+1}, 0^{n-r-1})$ to $\mathbf{b}$. The first assertion follows

first from repeated application of Lemma 3.3(iii) to $\mathbf{a}$ to give a path from $\mathbf{a}$ to an edge whose first r entries are non-zero and the last $n-r$ entries are zero. A path from this vertex to $(1^r, 0^{n-r})$ exists from Lemma 3.2.

An exactly analogous argument using repeated applications of Lemmas 3.3(iv) and 3.2 shows that there is a directed path from $(1^{r+1}, 0^{n-r-1})$ to $\mathbf{b}$. That is, to complete the proof it remains to show that there is a directed path in $H_k^{r,r+1}(n-1)$ from $(1^r, 0^{n-r})$ to $(1^{r+1}, 0^{n-r-1})$; but this is immediate since they differ in only one digit and hence the result follows from Lemma 2.1.

□

Remark 3.3. A simpler somewhat less formal approach to proving part (i) of the above theorem is to observe that the n-tuples corresponding to connected edges in $H_k^{r,r}(n-1)$ must all have the same 'pattern' of occurrences of zeros, up to cyclic shift. This follows from Lemma 3.2. Thus, since $H_k^{r,r}(n-1)$ contains n-tuples with different 'patterns' of zeros unless $r \in \{0, 1, n-1, n\}$, the result follows.

Theorem 3.4 has the following immediate corollary.

Corollary 3.5. *Suppose $k \geq 2$, $n \geq 2$ and $0 \leq r < s \leq n$. Then $H_k^{r,s}(n-1)$ is Eulerian.*

Proof. The fact that $H_k^{r,s}(n-1)$ is balanced is immediate from Lemma 2.1. We know that the set of edges in $H_k^{r,s}(n-1)$ is the union of the sets of edges in $H_k^{t,t+1}(n-1)$ for every t satisfying $r \leq t < s$, and $H_k^{t,t+1}(n-1)$ is strongly connected from Theorem 3.4(ii). So suppose $\mathbf{a}_t$ and $\mathbf{a}_u$ are edges in $H_k^{r,s}(n-1)$ of weights t and u respectively. Without loss of generality suppose $t \leq u$. If $u = t$ or $u = t+1$ then both $\mathbf{a}_t$ and $\mathbf{a}_u$ are in $H_k^{t,t+1}(n-1)$, and hence there is a directed path from $\mathbf{a}_t$ to $\mathbf{a}_u$. So suppose $u > t+1$. Then let $\mathbf{a}_{t+1}, \mathbf{a}_{t+2}, \ldots, \mathbf{a}_{u-1}$ be arbitrary edges of weights $t+1, t+2, \ldots, u-1$ respectively. Then since $H_k^{i,i+1}(n-1)$ is strongly connected for every i, there exist directed paths in $H_k^{r,s}(n-1)$ from $\mathbf{a}_i$ to $\mathbf{a}_{i+1}$ for every i $(t \leq i < u)$, and hence from $\mathbf{a}_t$ to $\mathbf{a}_u$. It follows that $H_k^{r,s}(n-1)$ is strongly connected. □

Example 3.1. We give examples of cut-down de Bruijn sequences arising from Eulerian circuits in subgraphs $H_k^{r,s}(n-1)$; in each case one period of the sequence is shown.

(i) Let $k = n = 3$ and $r = s = 2 = n-1$. By Theorem 3.4(i), $H_3^{2,2}(2)$ is Eulerian, and it has $\binom{3}{2}2^2 = 12$ edges. The sequence

$$(101202102201)$$

of period 12 contains every ternary 3-tuple of Hamming weight 2 exactly once in a period.

(ii) Let $k = n = 3$, $r = 1$ and $s = 2$. By Theorem 3.4(ii), $H_3^{1,2}(2)$ is Eulerian, and it has $\binom{3}{1}2 + \binom{3}{2}2^2 = 18$ edges. The sequence

$$(100201101202102200)$$

of period 18 contains every ternary 3-tuple of Hamming weight 1 or 2 exactly once in a period.

(iii) Let $k = 3$, $n = 4$, $r = 1$ and $s = 2$. Then $H_3^{1,2}(3)$ has $\binom{4}{1}2 + \binom{4}{2}2^2 = 32$ edges, and the sequence

$$(10002010102020011001200210022000)$$

of period 32 contains every ternary 4-tuple of Hamming weight 1 or 2 exactly once in a period.

(iv) Theorem 3.4(i) shows that $H_3^{2,2}(3)$ is not Eulerian. Indeed, as observed in Remark 3.3, the 24 edges of $H_3^{2,2}(3)$ fall into two connected components according to the cyclic pattern of their zeros: the 16 4-tuples whose two non-zero entries are cyclically adjacent (such as $(1, 2, 0, 0)$), and the 8 4-tuples whose two non-zero entries are not (such as $(1, 0, 2, 0)$).

4 Partitioning using digit sum

The second 'measure' for n-tuples that we consider is the digit sum, as in Definition 2.2. For this case we need the following.

Definition 4.1 (The digit sum subgraph)**.** Suppose $k \geq 2$, $n \geq 2$ and $0 \leq r \leq s \leq n(k-1)$. We write $S_k^{r,s}(n-1)$ for the subgraph of $B_k(n-1)$ consisting of the edges corresponding to n-tuples $\mathbf{a}$ satisfying $r \leq w_s(\mathbf{a}) \leq s$.

A special case of the problem we address here has been previously considered for this function.

Result 4.1 ([11])**.** *Suppose $k \geq 2$ and $n \geq 2$. Then $S_k^{0,s}(n-1)$ and $S_k^{s,n(k-1)}(n-1)$ are Eulerian, for every s ($0 \leq s \leq n(k-1)$).*

Remark 4.1. It is interesting to observe that Result 4.1 is briefly stated in earlier work by Sawada et al. [28, Section 3.3.1]. Using the notation of Section 1.3, the set of edges in $S_k^{s,n(k-1)}(n-1)$ is in $\mathcal{C}(n,k)$ for every s, $0 \leq s \leq n(k-1)$, since replacing entries of an n-tuple with $k-1$ can never decrease the digit sum. Hence, by Result 1.1, $S_k^{s,n(k-1)}(n-1)$ is Eulerian. The fact that $S_k^{0,s}(n-1)$ is Eulerian follows by mapping any symbol i to $k-1-i$, which is an isomorphism of the de Bruijn graph.

In fact a much more general result can be established. We first observe the following.

Lemma 4.2. *Suppose $k \geq 2$, $n \geq 1$ and $0 \leq s \leq n(k-1)$. Then the number of edges in $S_k^{s,s}(n-1)$ is*

$$N_k(n,s) = \sum_{j=0}^{\lfloor s/k \rfloor} (-1)^j \binom{n}{j} \binom{s-jk+n-1}{n-1}.$$

Equivalently, $N_k(n,s)$ is the coefficient of x^s in $(1+x+\cdots+x^{k-1})^n = \left(\frac{1-x^k}{1-x}\right)^n$.

Proof. Let U be the set of n-tuples of non-negative integers (not necessarily from $[0,k-1]$) whose digit sum equals s. A standard 'stars-and-bars' argument gives $|U| = \binom{s+n-1}{n-1}$. For $i \in \{1,2,\ldots,n\}$, let $A_i = \{(a_1,\ldots,a_n) \in U : a_i \geq k\}$. The tuples we wish to count are exactly those in $U - \bigcup_{i=1}^n A_i$.

Fix $J \subseteq \{1,2,\ldots,n\}$ with $|J| = j$. The mapping that subtracts k from each coordinate indexed by J is a bijection from $\bigcap_{i\in J} A_i$ to the set of n-tuples of non-negative integers with sum $s-jk$. Hence

$$\Big|\bigcap_{i\in J} A_i\Big| = \binom{s-jk+n-1}{n-1}$$

when $jk \leq s$, and the intersection is empty otherwise. By inclusion-exclusion,

$$N_k(n,s) = \sum_{J\subseteq\{1,\ldots,n\}} (-1)^{|J|} \Big|\bigcap_{i\in J} A_i\Big| = \sum_{j=0}^{\lfloor s/k \rfloor} (-1)^j \binom{n}{j} \binom{s-jk+n-1}{n-1}.$$

For the generating-function form, note that $N_k(n,s)$ is the number of ways of choosing one term x^{a_i} from each of the n factors of $(1+x+\cdots+x^{k-1})^n$ so that the exponents sum to s. That is, $N_k(n,s) = [x^s](1+x+\cdots+x^{k-1})^n$. Expanding $(1-x^k)^n(1-x)^{-n}$ by the binomial theorem gives the same sum. □

Lemma 4.3. *Suppose $k \geq 2$, $n \geq 2$, $0 \leq r \leq n(k-1)$ and $(a_1,a_2,\ldots,a_{n-1})$ is a vertex in $S_k^{r,r}(n-1)$. If $(a_0,a_1,\ldots,a_{n-1})$ is an incoming edge to this vertex then $(a_1,a_2,\ldots,a_n)$ is an outgoing edge if and only if $a_0 = a_n$.*

Proof. This is immediate since all edges in $S_k^{r,r}(n-1)$ correspond to n-tuples with digit sum r. □

Since $w_s(\mathbf{a}) = w_h(\mathbf{a})$ if $k = 2$, Theorem 3.4 applies in this case. Thus we restrict our attention here to the case $k > 2$. Before giving the main result we need the following key lemma.

Lemma 4.4. *Suppose $n \geq 2$, $k > 2$ and $0 \leq r < n(k-1)$.*

(i) If $\mathbf{a}$ is an edge (i.e. an n-tuple) in $S_k^{r,r+1}(n-1)$, then there is a directed path in $S_k^{r,r+1}(n-1)$ from $\mathbf{a}$ to an edge $\mathbf{a}'$ satisfying $w_s(\mathbf{a}') = r$.

(ii) If $\mathbf{b}$ is an edge (i.e. an n-tuple) in $S_k^{r,r+1}(n-1)$, then there is a directed path in $S_k^{r,r+1}(n-1)$ from an edge $\mathbf{b}'$ satisfying $w_s(\mathbf{b}') = r+1$ to $\mathbf{b}$.

(iii) If $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ is an edge (i.e. an n-tuple) in $S_k^{r,r+1}(n-1)$ such that $w_s(\mathbf{a}) = r$, $a_i < k-1$ and $a_{i+1} > 0$, for some i $(0 \leq i < n-1)$, then there is a directed path in $S_k^{r,r+1}(n-1)$ from $\mathbf{a}$ to

$$(a_0, a_1, \ldots, a_{i-1}, a_i + 1, a_{i+1} - 1, a_{i+2}, \ldots, a_{n-1}),$$

i.e. $\mathbf{a}$ modified so that a_i is increased by 1 and a_{i+1} decreased by 1.

(iv) If $\mathbf{b} = (b_0, b_1, \ldots, b_{n-1})$ is an edge (i.e. an n-tuple) in $S_k^{r,r+1}(n-1)$ such that $w_s(\mathbf{b}) = r+1$, $b_i < k-1$ and $b_{i+1} > 0$, for some i $(0 \leq i < n-1)$, then there is a directed path in $S_k^{r,r+1}(n-1)$ from

$$(b_0, b_1, \ldots, b_{i-1}, b_i + 1, b_{i+1} - 1, b_{i+2}, \ldots, b_{n-1})$$

i.e. $\mathbf{b}$ modified so that b_i is increased by 1 and b_{i+1} decreased by 1, to $\mathbf{b}$.

Proof. (i) If $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ satisfies $w_s(\mathbf{a}) = r+1$, then suppose $a_i \neq 0$ for some i (which must exist since $w_s(\mathbf{a}) > 0$). Now using cyclic shifting there is a directed path in $S_k^{r,r+1}(n-1)$ from $\mathbf{a}$ to $(a_i, a_{i+1}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{i-1})$. Because $w_s(\mathbf{a}) = r+1$, the edge $(a_{i+1}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{i-1}, a_i - 1)$ is an edge in $S_k^{r,r+1}(n-1)$ with digit sum r, and so a directed path exists in $S_k^{r,r+1}(n-1)$ from $\mathbf{a}$ to an edge with digit sum r. Alternatively, if $w_s(\mathbf{a}) = r$, then simply set $\mathbf{a} = \mathbf{a}'$. Either way the result holds.

(ii) If $\mathbf{b} = (b_0, b_1, \ldots, b_{n-1})$ satisfies $w_s(\mathbf{b}) = r < n(k-1)$, then there is at least one value b_i such that $b_i < k-1$. As in the previous case, using cyclic shifting there is a directed path in $S_k^{r,r+1}(n-1)$ from $(b_i, b_{i+1}, \ldots, b_{n-1}, b_0, b_1, \ldots, b_{i-1})$ to $\mathbf{b}$. Because $w_s(\mathbf{b}) = r$, the edge $(b_i + 1, b_{i+1}, \ldots, b_{n-1}, b_0, b_1, \ldots, b_{i-1})$ is an edge in $S_k^{r,r+1}(n-1)$ with digit sum $r+1$, and the desired directed path exists. Alternatively, if $w_s(\mathbf{b}) = r+1$, then simply set $\mathbf{b} = \mathbf{b}'$. Either way the result holds.

(iii) There is clearly a directed path from $\mathbf{a}$ to $(a_i, a_{i+1}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{i-1})$ in $S_k^{r,r+1}(n-1)$ simply from cyclic shifting. Since $\mathbf{a}$ has digit sum r, the n-tuple

$$(a_{i+1}, a_{i+2}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{i-1}, a_i + 1)$$

has digit sum $r+1$ and hence is in $S_k^{r,r+1}(n-1)$. It follows that

$$(a_{i+2}, a_{i+3}, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{i-1}, a_i + 1, a_{i+1} - 1)$$

has digit sum r and hence is also in $S_k^{r,r+1}(n-1)$. Finally, there is a directed path from this latter n-tuple to

$$(a_0, a_1, \ldots, a_{i-1}, a_i + 1, a_{i+1} - 1, a_{i+2}, \ldots, a_{n-1}),$$

via cyclic shifting and so the desired directed path exists.

(iv) There is a directed path from $(b_{i+2}, b_{i+3}, \ldots, b_{n-1}, b_0, b_1, \ldots, b_{i+1})$ to $\mathbf{b}$ in $S_k^{r,r+1}(n-1)$ using cyclic shifting. Because $\mathbf{b}$ has digit sum $r+1$, the n-tuple

$$(b_{i+1} - 1, b_{i+2}, \ldots, b_{n-1}, b_0, b_1, \ldots, b_i)$$

has digit sum r and hence is in $S_k^{r,r+1}(n-1)$. Hence

$$(b_i + 1, b_{i+1} - 1, b_{i+2}, \ldots, b_{n-1}, b_0, b_1, \ldots, b_{i-1})$$

has digit sum $r+1$ and so is likewise in $S_k^{r,r+1}(n-1)$. Finally, there is a directed path from

$$(b_0, b_1, \ldots, b_{i-1}, b_i + 1, b_{i+1} - 1, b_{i+2}, \ldots, b_{n-1}),$$

to this latter n-tuple via cyclic shifting and so the desired directed path exists. This completes the proof.

□

We can now give the following result.

Theorem 4.5. *Suppose $n \geq 2$ and $k > 2$. Then:*

(i) $S_k^{r,r}(n-1)$ is Eulerian if and only if $r \in \{0, 1, (k-1)n-1, (k-1)n\}$;

(ii) $S_k^{r,r+1}(n-1)$ is Eulerian for every r $(0 \leq r < (k-1)n)$.

Proof. (i) If $r = 0$ or $r = (k-1)n$ then the only n-tuple with digit sum r is 0^n or $(k-1)^n$, respectively. The result follows. Somewhat similarly, the only n-tuple with digit sum $r = 1$ consists of $n-1$ zeros and a single 1 symbol. Clearly this set of n n-tuples can be arranged in a single cycle — a necklace. A precisely analogous argument holds for $r = (k-1)n - 1$.

Now suppose $1 < r < (k-1)n - 1$. By Lemma 4.3, the only cycles in $S_k^{r,r}(n-1)$ consist of necklaces, and thus have period dividing n. However, if $1 < r < (k-1)n - 1$ then $S_k^{r,r}(n-1)$ will clearly contain more than n edges (n-tuples), and hence it contains more than one necklace. However, two necklaces in $S_k^{r,r}(n-1)$ cannot share a vertex since there is only at most one edge of weight r outgoing from (and incoming to) any vertex. The desired result follows.

(ii) Just as previously, because of Lemma 2.1, establishing the Eulerian property just requires us to show that the subgraphs are strongly connected. Suppose $\mathbf{a}$ and $\mathbf{b}$ are edges in $S_k^{r,r+1}(n-1)$. Then, by Lemma 4.4(i) and (ii), there are directed paths in $S_k^{r,r+1}(n-1)$ from $\mathbf{a}$ to $\mathbf{a}'$ and from $\mathbf{b}'$ to $\mathbf{b}$, where $\mathbf{a}'$ has digit sum r and $\mathbf{b}'$ has digit sum $r+1$. So to establish the result we simply need to show there is a path in $S_k^{r,r+1}(n-1)$ from any n-tuple with digit sum r to any n-tuple with digit sum $r+1$.

Suppose $\mathbf{a}$ has digit sum r and $\mathbf{b}$ has digit sum $r+1$, where $r = s(k-1)+t$, $s \geq 0$, and $0 \leq t < k-1$. Then we claim there are directed paths in $S_k^{r,r+1}(n-1)$ from $\mathbf{a}$ to $((k-1)^s, t, 0^{n-s-1})$, and from $((k-1)^s, t+1, 0^{n-s-1})$ to $\mathbf{b}$. The first assertion follows from repeated application of Lemma 4.4(iii) to $\mathbf{a}$ until an n-tuple $(c_0, c_1, \ldots, c_{n-1})$ of digit sum r is reached for which there is no i such that $c_i < k-1$ and $c_{i+1} > 0$; this process terminates since each step increases the tuple lexicographically. It follows that $\mathbf{c} = ((k-1)^s, t, 0^{n-s-1})$, and the first claim is established. An exactly analogous argument using repeated applications of Lemma 4.4(iv) shows that there is a directed path from $((k-1)^s, t+1, 0^{n-s-1})$ to $\mathbf{b}$. That is, to complete the proof it remains to show that there is a directed path in $S_k^{r,r+1}(n-1)$ from $((k-1)^s, t, 0^{n-s-1})$ to $((k-1)^s, t+1, 0^{n-s-1})$; but this is immediate since they differ in only one digit, and hence the result follows from Lemma 2.1.

□

Clearly Result 4.1 follows from Theorem 4.5 as a special case.

Analogously to the Hamming weight case, we have the following simple corollary, whose proof is immediate using the same arguments as for Corollary 3.5.

Corollary 4.6. *Suppose $n \geq 2$, $k > 2$ and $0 \leq r < s \leq (k-1)n$. Then $S_k^{r,s}(n-1)$ is Eulerian.*

Example 4.1. We give examples of cut-down de Bruijn sequences arising from Eulerian circuits in subgraphs $S_k^{r,s}(n-1)$; in each case one period of the sequence is shown.

(i) Let $k = n = 3$. By Theorem 4.5(i), $S_3^{2,2}(2)$ is not Eulerian. Indeed, its six edges form the two necklaces $[0,0,2]$ and $[0,1,1]$, each of which is a directed cycle of length 3 in $B_3(2)$, and these two cycles are disjoint.

(ii) By Theorem 4.5(ii), $S_3^{2,3}(2)$ is Eulerian. It has $6+7 = 13$ edges, and the sequence

(2011102101200)

of period 13 contains every ternary 3-tuple with digit sum 2 or 3 exactly once in a period.

(iii) Let $k = 4$ and $n = 3$. Then $S_4^{3,4}(2)$ has $10 + 12 = 22$ edges, and the sequence

$$(3012021112103102201300)$$

of period 22 contains every 4-ary 3-tuple with digit sum 3 or 4 exactly once in a period.

(iv) Let $k = 3$ and $n = 4$. Then $S_3^{3,4}(3)$ has $16 + 19 = 35$ edges, and the sequence

$$(20021011110202011201021101210022001)$$

of period 35 contains every ternary 4-tuple with digit sum 3 or 4 exactly once in a period.

5 Partitioning using pseudoweight

The third 'measure' for n-tuples that we consider is the pseudoweight, as in Definition 2.3. If $k = 2$ then $w_p(0) = w_p(1) = 1$ and the pseudoweight function is trivial and uninteresting. Thus, in the remainder of this discussion we restrict our attention to $k > 2$. We need the following.

Definition 5.1 (The pseudoweight subgraph)**.** Suppose $k > 2$, $n \geq 2$ and $n \leq r \leq s \leq (k-1)n$ (where $r \in \mathbb{Z}$ if k is even and $r = t/2$ for some $t \in \mathbb{Z}$ if k is odd). We write $P_k^{r,s}(n-1)$ for the subgraph of $B_k(n-1)$ consisting of the edges corresponding to n-tuples $\mathbf{a}$ satisfying $r \leq w_p(\mathbf{a}) \leq s$.

Exactly as for digit sum, special cases of the problem we address here have been previously considered for this function. The discussion following [18, Theorem 3.8] establishes the following result.

Result 5.1 ([18])**.** *Suppose $k > 2$ and $n \geq 2$. Then $P_k^{n,s}(n-1)$ is Eulerian, where*

$$s = \begin{cases} \frac{nk-1}{2} & \text{if } k \text{ is odd} \\ \frac{nk-2}{2} & \text{if } k \text{ is even} \end{cases}$$

Remark 5.1. Since $\frac{nk}{2}$ is the average value of the function w_p when calculated over all n-tuples, this simply says that the subgraph of $B_k(n-1)$ consisting of all n-tuples with pseudoweight less than the average value is Eulerian.

Remark 5.2. In fact the earlier work of Sawada et al. [28] can be used to give a stronger result than Result 5.1. Using the notation of Section 1.3, the set of edges in $P_k^{s,n(k-1)}(n-1)$ is in $\mathcal{C}(n,k)$ for every s, $n \leq s \leq n(k-1)$, since replacing entries of an n-tuple with $k-1$ can never decrease the

pseudoweight. Hence, by Result 1.1, $P_k^{s,n(k-1)}(n-1)$ is Eulerian. Mapping every symbol i to $k-i$ (modulo k) gives an isomorphism of the de Bruijn graph which maps $P_k^{s,n(k-1)}(n-1)$ to $P_k^{n,nk-s}(n-1)$, since $w_p(k-i) = k-w_p(i)$ for every i. Hence $P_k^{n,s}(n-1)$ is also Eulerian for every s satisfying $n \leq s \leq n(k-1)$.

First, we have the following. Suppose $k > 2$, $n \geq 1$ and $0 \leq i \leq n(k-1)$ (where the range includes values of the form $u/2$ where u is an integer). Following [18], let the number of edges in $P_k^{i,i}(n-1)$ be denoted by $r_{k,n,i}$. Then we have the following result.

Result 5.2 (Lemma 3.10 of [18])**.** *Suppose $k > 2$ and $n \geq 1$.*

(i) If k is even, $r_{k,1,i} = 1$ for $1 \leq i \leq k-1$, $i \neq k/2$, and $r_{k,1,k/2} = 2$.

(ii) If k is odd, $r_{k,1,i} = 1$ for $1 \leq i \leq k-1$, and $r_{k,1,k/2} = 1$.

(iii) If k is even, $\sum_{i=n}^{n(k-1)} r_{k,n,i} = k^n$.

(iv) If k is odd, $\sum_{j=0}^{2n(k-2)} r_{k,n,n+j/2} = k^n$.

(v) If $n > 1$, $r_{k,n,s} = \sum_{i=1}^{k-1} r_{k,n-1,s-i} + r_{k,n-1,s-k/2}$.

Result 5.2(v) gives a simple recursion that, combined with (i) and (ii), enable the number of edges in $P_k^{i,i}(n-1)$ to be readily computed for any i.

Lemma 5.3. *Suppose $k > 2$, $n \geq 2$, $n \leq r \leq n(k-1)$ and $(a_1, a_2, \ldots, a_{n-1})$ is a vertex in $P_k^{r,r}(n-1)$. If $(a_0, a_1, \ldots, a_{n-1})$ is an incoming edge to this vertex then $(a_1, a_2, \ldots, a_n)$ is an outgoing edge if and only if:*

$$a_n = \begin{cases} 0 \text{ or } k/2 & \text{if } k \text{ is even and } a_0 = 0 \text{ or } a_0 = k/2 \\ a_0 & \text{otherwise} \end{cases}$$

Proof. Since both $(a_0, a_1, \ldots, a_{n-1})$ and $(a_1, a_2, \ldots, a_n)$ have the same pseudoweight, it follows that $f(a_0) = f(a_n)$, where f is as defined in Definition 2.3. Hence, unless k is even and $f(a_0) = f(a_n) = k/2$, this automatically implies $a_0 = a_n$. Conversely, if $f(a_0) = f(a_n)$ then $(a_0, a_1, \ldots, a_{n-1})$ and $(a_1, a_2, \ldots, a_n)$ have the same pseudoweight, and the result follows. □

The analysis for this function is somewhat more complex. However, we do have a result similar to Theorems 3.4(i) and 4.5.

Theorem 5.4. *Suppose $n \geq 2$, $k > 2$, $n \leq r \leq n(k-1)$ (where $r \in \mathbb{Z}$ if k is even and $r = t/2$ for some $t \in \mathbb{Z}$ if k is odd), and $P_k^{r,r}(n-1)$ is non-empty. Then $P_k^{r,r}(n-1)$ is Eulerian if and only if*

(i) $r \in \{n, (k-1)n\}$,

(ii) $r \in \{n+1, (k-1)n-1\}$ *and* $k \neq 3$,

(iii) $r \in \{n-1+k/2, (k-1)(n-1)+k/2\}$ *and* k *odd,*

(iv) $r \in \{i+k/2 : 1 \leq i \leq k-1\}$, $n = 2$ *and* k *odd.*

Remark 5.3. Note that when $n = 2$, the cases $i = 1$ and $i = k-1$ of (iv) coincide with (iii). Also, the requirement that $P_k^{r,r}(n-1)$ is non-empty only excludes the case where r is not an integer and $n < r < n-1+k/2$ or $(k-1)(n-1)+k/2 < r < (k-1)n$, which can only happen when $k \geq 5$ is odd.

Proof. Because of Lemma 2.1, to show the Eulerian property holds it suffices to show that the relevant subgraph is strongly connected.

(i) $P_k^{n,n}(n-1)$ consists of a single edge, namely 1^n — a loop from the vertex 1^{n-1} to itself; this is clearly Eulerian. Similarly, the subgraph $P_k^{(k-1)n,(k-1)n}(n-1)$ consists of the single edge $(k-1)^n$ and thus is also Eulerian.

(ii) $P_4^{n+1,n+1}(n-1)$ contains all the n-tuples in the two necklaces $[1^{n-1}, 2]$ and $[1^{n-1}, 0]$. These two necklaces share a vertex, namely 1^{n-1}, and hence the subgraph is strongly connected. If $k > 4$, then $P_k^{n+1,n+1}(n-1)$ contains only a single necklace, namely $[1^{n-1}, 2]$, and thus the subgraph is strongly connected. Analogous arguments hold for $P_4^{3n-1,3n-1}(n-1)$, which contains the necklaces $[3^{n-1}, 2]$ and $[3^{n-1}, 0]$, and $P_k^{n(k-1)-1,n(k-1)-1}(n-1)$, which contains the single necklace $[(k-1)^{n-1}, k-2]$.

(iii) If k is odd then $n-1+k/2$ is not an integer, and hence the only necklace in $P_k^{n-1+k/2,n-1+k/2}(n-1)$ is $[1^{n-1}, 0]$, and so $P_k^{n-1+k/2,n-1+k/2}(n-1)$ is Eulerian. Similarly, $(k-1)(n-1)+k/2$ is not an integer, and hence the only necklace in $P_k^{(k-1)(n-1)+k/2,(k-1)(n-1)+k/2}(n-1)$ is $[(k-1)^{n-1}, 0]$ and so $P_k^{(k-1)(n-1)+k/2,(k-1)(n-1)+k/2}(n-1)$ is Eulerian.

(iv) If $n = 2$ and k is odd then when $r = i+k/2$ with $1 \leq i \leq k-1$ $P_k^{i+k/2,i+k/2}(n-1)$ only contains the edges $(0,i), (i,0)$, a necklace of period 2, and so $P_k^{i+k/2,i+k/2}(n-1)$ is Eulerian. This holds since if a tuple has a non-integer weight then it must contain an odd number of zeros, i.e. a single zero when $n = 2$.

It remains to show that in all other cases $P_k^{r,r}(n-1)$ is not Eulerian. First consider the special case $k = 3$ and $r \in \{n+1, (k-1)n-1\}$. Observe that $P_3^{n+1,n+1}$ contains two necklaces: $[1^{n-1}, 2]$ and $[1^{n-2}, 0, 0]$. These clearly

do not share a vertex, and hence $P_3^{n+1,n+1}$ is not Eulerian. Similarly, $P_3^{2n-1,2n-1}(n-1)$ contains the two necklaces $[2^{n-1},1]$ and $[2^{n-2},0,0]$. These clearly do not share a vertex, and hence $P_3^{2n-1,2n-1}(n-1)$ is not Eulerian.

Next, suppose k is odd. By Lemma 5.3 an edge $(a_0,\ldots,a_{n-1})$ of $P_k^{r,r}(n-1)$ has only one incoming edge and only one outgoing edge in $P_k^{r,r}(n-1)$, namely $(a_{n-1},a_0,\ldots,a_{n-2})$ and $(a_1,\ldots,a_{n-1},a_0)$ respectively. It follows that each connected component of $P_k^{r,r}(n-1)$ is a single necklace, and so we need only show that $P_k^{r,r}(n-1)$ contains more than one necklace in the remaining cases. Since $P_k^{r,r}(n-1)$ is non-empty, these are: $r\in\mathbb{Z}$ and $n+2\le r\le(k-1)n-2$; and $r\notin\mathbb{Z}$, $n\ge 3$ and $n-1+k/2<r<(k-1)(n-1)+k/2$.

First suppose $r\in\mathbb{Z}$ and $n+2\le r\le(k-1)n-2$. If $k=3$ (and so $n\ge 4$), let $\mathbf{b}$ be an $(n-2)$-tuple with entries from $\{1,2\}$ and digit sum $r-3$, which exists since $n-2\le r-3\le 2(n-2)$. Then $(\mathbf{b},0,0)$ and $(\mathbf{b},1,2)$ are both edges of $P_3^{r,r}(n-1)$, since $w_p(0,0)=3=w_p(1,2)$, and they lie in different necklaces since only one of them contains a zero. If $k\ge 5$, then since

$$[n+2,(k-1)n-2]=[n-2,(k-1)(n-2)]+[4,2k-4],$$

there exist an integer u with $4\le u\le 2k-4$ and a zero-free $(n-2)$-tuple $\mathbf{b}$ with digit sum $r-u$ (where $\mathbf{b}$ is empty if $n=2$). If $u\le k$, the edges $(\mathbf{b},1,u-1)$ and $(\mathbf{b},2,u-2)$ both lie in $P_k^{r,r}(n-1)$; if $u>k$, the edges $(\mathbf{b},k-1,u-k+1)$ and $(\mathbf{b},k-2,u-k+2)$ do. In either case the two edges are zero-free, so their pseudoweight equals their digit sum, and they have different multisets of entries (since $u\ne 3$ and $u\ne 2k-3$), so they lie in different necklaces.

Now suppose $r\notin\mathbb{Z}$, $n\ge 3$ and $n-1+k/2<r<(k-1)(n-1)+k/2$, and let $m=r-k/2$, so that $m\in\mathbb{Z}$ and $n-1<m<(k-1)(n-1)$. There are at least two distinct zero-free $(n-1)$-tuples with digit sum m: if $\mathbf{c}$ is one such tuple whose entries are not all equal, then a non-trivial cyclic shift of $\mathbf{c}$ is another; otherwise $\mathbf{c}=x^{n-1}$ with $1<x<k-1$, and then $(x-1,x+1,x^{n-3})$ is another. If $\mathbf{c}$ and $\mathbf{c}'$ are two such tuples, then $(0,\mathbf{c})$ and $(0,\mathbf{c}')$ are edges of $P_k^{r,r}(n-1)$. Since each contains exactly one zero, each of their necklaces contains exactly one n-tuple beginning with 0, and hence they lie in different necklaces.

In every case $P_k^{r,r}(n-1)$ contains more than one necklace, and hence is not strongly connected.

Finally, suppose k is even and $n+1<r<(k-1)n-1$. Suppose $r=n+s(k-2)+t-1$, where $1\le t<k-1$ and $0\le s<n$. Then there exists an edge in $P_k^{r,r}(n-1)$ of the form $((k-1)^s,t,1^{n-s-1})$. No vertex in this necklace will have more than one incoming edge or outgoing edge in $P_k^{r,r}(n-1)$ except in the case $t=k/2$. If $t=k/2$ then the necklace $[(k-1)^s,0,1^{n-s-1}]$ will share a vertex with the necklace $[(k-1)^s,t,1^{n-s-1}]$ but neither necklace will share a vertex with any other edges in $P_k^{r,r}(n-1)$.

We next show that $P_k^{r,r}(n-1)$ will always contain another necklace not equal to either $[(k-1)^s, t, 1^{n-s-1}]$ (or $[(k-1)^s, 0, 1^{n-s-1}]$ if $t = k/2$), establishing that $P_k^{r,r}(n-1)$ is not strongly connected. We need to consider three cases.

- First suppose $s = 0$, i.e. there exists an edge in $P_k^{r,r}(n-1)$ of the form $(t, 1^{n-1})$, where $t > 2$ since $r > n+1$. Then there will also exist an edge of the form $(t-1, 2, 1^{n-2})$ in $P_k^{r,r}(n-1)$, since $n \geq 2$, i.e. another necklace.
- Next suppose $s > 0$ and $t < k-2$; then, since $k \geq 4$, there exists a further edge of the form $((k-1)^{s-1}, k-2, t+1, 1^{n-s-1})$ in $P_k^{r,r}(n-1)$.
- Finally suppose $s > 0$ and $t = k-2$. Now, since $r < (k-1)n-1$, $n-s-1 > 0$. Then the edge mentioned above takes the form $((k-1)^s, k-2, 1^{n-s-1})$. An additional edge is then $((k-1)^s, k-3, 2, 1^{n-s-2})$.

The result follows. □

The k even case is somewhat simpler than the k odd case, since if k is even then $w_p(\mathbf{a}) \in \mathbb{Z}$ for any k-ary n-tuple $\mathbf{a}$. We first need the following preliminary lemma.

Lemma 5.5. *Suppose $n \geq 2$, $k > 2$ is even, and $n \leq r \leq n(k-1)$. Suppose $\mathbf{a}$ is an edge (i.e. a k-ary n-tuple) in $P_k^{r,r}(n-1)$ and $\bar{\mathbf{a}}$ is equal to $\mathbf{a}$ with every 0 changed to $k/2$. Then there exist directed paths in $P_k^{r,r}(n-1)$ from $\mathbf{a}$ to $\bar{\mathbf{a}}$ and from $\bar{\mathbf{a}}$ to $\mathbf{a}$.*

Proof. The result follows using repeated applications of Lemma 2.1. That is, to obtain a directed path from $\mathbf{a}$ to $\bar{\mathbf{a}}$, simply use Lemma 2.1 to obtain a directed path from $\mathbf{a}$ to an edge $\mathbf{a}'$ equal to $\mathbf{a}$ with a single 0 changed to $k/2$, and then a directed path from $\mathbf{a}'$ to an edge $\mathbf{a}''$ equal to $\mathbf{a}'$ with another 0 changed to $k/2$, and so on until $\bar{\mathbf{a}}$ is reached. For the path in the opposite direction simply change the relevant entries of $k/2$ to 0 one by one until $\mathbf{a}$ is reached. □

We can now give the main result.

Theorem 5.6. *Suppose $n \geq 2$ and $k > 2$ is even. Then $P_k^{r,r+1}(n-1)$ is Eulerian for every $r \in \mathbb{Z}$ satisfying $n \leq r < (k-1)n$.*

Proof. By Lemma 2.1 we need only show that $P_k^{r,r+1}(n-1)$ is strongly connected. By Lemma 5.5, every edge of $P_k^{r,r+1}(n-1)$ is joined in both directions by directed paths in $P_k^{r,r+1}(n-1)$ to an edge containing no zeros. Hence we need only show that there is a directed path in $P_k^{r,r+1}(n-1)$ between any two edges whose corresponding n-tuples contain no zeros.

Let Z be the subgraph of $P_k^{r,r+1}(n-1)$ whose edges are the zero-free n-tuples in $P_k^{r,r+1}(n-1)$. If $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ is zero-free then $w_p(\mathbf{a}) = \sum_{i=0}^{n-1} a_i$, and so the map

$$(a_0, a_1, \ldots, a_{n-1}) \mapsto (a_0 - 1, a_1 - 1, \ldots, a_{n-1} - 1)$$

is a bijection from the edges of Z to the $(k-1)$-ary n-tuples with digit sum $r-n$ or $r-n+1$. Since it is induced by a bijection on symbols, it preserves the de Bruijn adjacency, and so it is an isomorphism from Z to $S_{k-1}^{r-n,r-n+1}(n-1)$. Moreover $0 \leq r-n < (k-2)n$, which is precisely the range of r required for $S_{k-1}^{r,r+1}(n-1)$.

If $k > 4$ then $k-1 > 2$, and $S_{k-1}^{r-n,r-n+1}(n-1)$ is strongly connected by Theorem 4.5(ii). If $k = 4$ then $k-1 = 2$ and so the digit sum of a binary n-tuple equals its Hamming weight, so $S_2^{r-n,r-n+1}(n-1) = H_2^{r-n,r-n+1}(n-1)$, which is strongly connected by Theorem 3.4(ii). In either case Z is strongly connected, and the result follows. □

To analyse the k odd case we first need the following.

Lemma 5.7. *Suppose $n \geq 2$, $k > 2$ is odd, and $n \leq r \leq n(k-1)$, where $r = t/2$ for some $t \in \mathbb{Z}$. Then:*

(i) if $\mathbf{a}$ has pseudoweight r, there exists a directed path in $P_k^{r,r+0.5}(n-1)$ from $\mathbf{a}$ to $\mathbf{a}'$, where $\mathbf{a}'$ contains no entries equal to 0;

(ii) if $\mathbf{b}$ has pseudoweight $r+0.5$, there exists a directed path in $P_k^{r,r+0.5}(n-1)$ from $\mathbf{b}'$ to $\mathbf{b}$, where $\mathbf{b}'$ contains no entries equal to 0.

Proof. (i) First observe that, for any k-ary n-tuple $\mathbf{a}$ of pseudoweight s, replacing a single zero in $\mathbf{a}$ by $(k-1)/2$ results in a k-ary n-tuple of pseudoweight $s-0.5$, and replacing a single zero by $(k+1)/2$ results in a k-ary n-tuple of pseudoweight $s+0.5$. So, starting from $\mathbf{a}$ (of pseudoweight r), replace every 0 one by one with alternately $(k+1)/2$ and $(k-1)/2$ until an n-tuple is reached (which we label $\mathbf{a}'$) containing no 0 entries. In every case the pseudoweight of the result will be either r or $r+0.5$, and hence by Lemma 2.1 there is a directed path from the starting n-tuple to the modified n-tuple in $P_k^{r,r+0.5}(n-1)$. Joining these paths together yields the desired directed path from $\mathbf{a}$ to $\mathbf{a}'$.

(ii) The argument for (i) works in reverse, replacing every 0 entry in $\mathbf{b}$ with alternately $(k-1)/2$ and $(k+1)/2$, yielding a directed path in $P_k^{r,r+0.5}(n-1)$ starting at an n-tuple $\mathbf{b}'$ containing no entries equal to 0 and ending in $\mathbf{b}$.

□

Remark 5.4. Of course, if the pseudoweight of an n-tuple is less than $n + (k-2)/2$ or more than $n(k-1) - (k-2)/2$ then an n-tuple cannot contain any zeros, so the Lemma holds trivially.

We can now give a companion result to Theorem 5.6.

Theorem 5.8. *Suppose $n \geq 2$ and $k > 2$ is odd. Then $P_k^{r,r+1}(n-1)$ is Eulerian for every $r \in \mathbb{Z}$ satisfying $n \leq r < (k-1)n$.*

Proof. As previously, by Lemma 2.1 we need only show that $P_k^{r,r+1}(n-1)$ is strongly connected. Since k is odd, every edge $\mathbf{a}$ of $P_k^{r,r+1}(n-1)$ satisfies $w_p(\mathbf{a}) \in \{r, r+0.5, r+1\}$. Note also that a zero-free n-tuple has integer pseudoweight, so any zero-free edge of $P_k^{r,r+1}(n-1)$ has pseudoweight r or $r+1$.

We first show that from every edge $\mathbf{a}$ of $P_k^{r,r+1}(n-1)$ there is a directed path in $P_k^{r,r+1}(n-1)$ to a zero-free edge $\mathbf{a}^*$. If $\mathbf{a}$ is zero-free, set $\mathbf{a}^* = \mathbf{a}$. Otherwise there are three cases.

- If $w_p(\mathbf{a}) = r$, then Lemma 5.7(i) gives the required path in $P_k^{r,r+0.5}(n-1) \subseteq P_k^{r,r+1}(n-1)$.
- If $w_p(\mathbf{a}) = r+0.5$, then Lemma 5.7(i), applied with $r+0.5$ in place of r, gives the required path in $P_k^{r+0.5,r+1}(n-1) \subseteq P_k^{r,r+1}(n-1)$.
- If $w_p(\mathbf{a}) = r+1$, let $\mathbf{a}'$ equal $\mathbf{a}$ with one zero replaced by $(k-1)/2$, so that $w_p(\mathbf{a}') = r+0.5$. Since $\mathbf{a}$ and $\mathbf{a}'$ differ in a single position, Lemma 2.1(i) gives a directed path from $\mathbf{a}$ to $\mathbf{a}'$ in $P_k^{r+0.5,r+1}(n-1)$. The previous case then gives a directed path from $\mathbf{a}'$ to a zero-free edge $\mathbf{a}^*$.

Symmetrically, we show that for every edge $\mathbf{b}$ of $P_k^{r,r+1}(n-1)$ there is a directed path in $P_k^{r,r+1}(n-1)$ from a zero-free edge $\mathbf{b}^*$ to $\mathbf{b}$. If $\mathbf{b}$ is zero-free, set $\mathbf{b}^* = \mathbf{b}$. Otherwise there are again three cases.

- If $w_p(\mathbf{b}) = r+1$, then Lemma 5.7(ii), applied with $r+0.5$ in place of r, gives the required path in $P_k^{r+0.5,r+1}(n-1)$.
- If $w_p(\mathbf{b}) = r+0.5$, then Lemma 5.7(ii) gives the required path in $P_k^{r,r+0.5}(n-1)$.
- If $w_p(\mathbf{b}) = r$, let $\mathbf{b}'$ equal $\mathbf{b}$ with one zero replaced by $(k+1)/2$, so that $w_p(\mathbf{b}') = r+0.5$. Lemma 2.1(i) gives a directed path from $\mathbf{b}'$ to $\mathbf{b}$ in $P_k^{r,r+0.5}(n-1)$. The previous case then gives a directed path from a zero-free edge $\mathbf{b}^*$ to $\mathbf{b}'$.

It remains to show that there is a directed path in $P_k^{r,r+1}(n-1)$ from $\mathbf{a}^*$ to $\mathbf{b}^*$. Let Z be the subgraph of $P_k^{r,r+1}(n-1)$ whose edges are its zero-free n-tuples. Exactly as in the proof of Theorem 5.6, subtracting one from every entry gives an isomorphism from Z to $S_{k-1}^{r-n,r-n+1}(n-1)$, where $0 \leq r-n < (k-2)n$. If $k > 3$ then $k-1 > 2$, and this graph is strongly connected by Theorem 4.5(ii). If $k = 3$ then $k-1 = 2$, and $S_2^{r-n,r-n+1}(n-1) = H_2^{r-n,r-n+1}(n-1)$ is strongly connected by Theorem 3.4(ii). Hence Z is strongly connected, so there is a directed path from $\mathbf{a}^*$ to $\mathbf{b}^*$, and the result follows. □

The separate results for k odd and k even (Theorems 5.6 and 5.8) can be combined to give the following, whose proof is immediate using the same arguments as for Corollary 3.5.

Corollary 5.9. *Suppose $n \geq 2$, $k > 2$ and $n \leq r < s \leq (k-1)n$, where $r, s \in \mathbb{Z}$. Then $P_k^{r,s}(n-1)$ is Eulerian.*

Remark 5.5. The requirement that $r, s \in \mathbb{Z}$ in Corollary 5.9 cannot in general be dropped when $k \geq 5$ is odd. For example, $P_5^{9/2,11/2}(1)$ is not strongly connected, since its edges $(1,4)$ and $(4,1)$ form a component on the vertices $\{1,4\}$ that is disjoint from its other edges.

Example 5.1. We give examples of cut-down de Bruijn sequences arising from Eulerian circuits in subgraphs $P_k^{r,s}(n-1)$; in each case one period of the sequence is shown. Recall that $f(0) = k/2$, so that 0 and $k/2$ contribute equally to the pseudoweight when k is even.

(i) Let $k = 4$, $n = 3$ and $r = s = 4 = n+1$. By Theorem 5.4(ii), $P_4^{4,4}(2)$ is Eulerian. Its six edges form the necklaces $[0,1,1]$ and $[1,1,2]$, and the sequence

$$(121101)$$

of period 6 contains each of these 3-tuples exactly once in a period.

(ii) For $k = 3$ the subgraph $P_3^{4,4}(2)$ is not Eulerian, as Theorem 5.4 requires $k \neq 3$ in case (ii). Here $f(0) = 3/2$, and the six edges of $P_3^{4,4}(2)$ form the necklaces $[0,0,1]$ and $[1,1,2]$, which are disjoint directed cycles in $B_3(2)$.

(iii) Let $k = 4$, $n = 3$, $r = 4$ and $s = 5$. By Theorem 5.6, $P_4^{4,5}(2)$ is Eulerian; it has $6 + 15 = 21$ edges, and the sequence

$$(101102113112012122100)$$

of period 21 contains every 4-ary 3-tuple of pseudoweight 4 or 5 exactly once in a period.

(iv) Let $k = 3$, $n = 3$, $r = 4$ and $s = 5$. By Theorem 5.8, $P_3^{4,5}(2)$ is Eulerian. Since k is odd it contains 3-tuples of pseudoweight 4, 9/2 and 5; there are 6, 7 and 6 of these respectively (note that $w_p(0,0,0) = 9/2$). The sequence

$$(01002012102112212 00)$$

of period 19 contains every ternary 3-tuple of pseudoweight between 4 and 5 exactly once in a period.

6 Concluding remarks

Perhaps the most pressing question is how to efficiently generate the universal sequences which are shown to exist in this paper. The results given here in every case show that certain subgraphs of the de Bruijn graph are Eulerian, and hence there exists a universal sequence containing all the n-tuples corresponding to edges in the relevant subgraph. However this does not, in itself, show how to construct an example of a universal sequence. Of course, as noted in Section 1, there are simple and relatively efficient algorithms for finding Eulerian circuits — and hence in our case a universal sequence — in any Eulerian graph.

However, for certain applications, the standard algorithms for finding circuits might not be appropriate. The issue of finding very efficient algorithms for generating circuits in specific subgraphs of the de Bruijn graph has been addressed by a number of authors.

- In 2013 Sawada et al. [29] described a method of efficiently constructing universal sequences containing all binary n-tuples with Hamming weight in the range $[c, d]$ for any c, d satisfying $0 \leq c < d \leq n$; a further result of this type is given in the 2014 paper of Sawada et al. [27].
- Recently, in 2026, Sawada et al. [25] gave an approach using 'concatenation trees' to construct a range of classes of universal sequences in a highly efficient way, including orientable sequences.
- Again in 2026, Campbell et al. [6] showed how to efficiently construct universal sequences for t-subsets and t-multisets.
- Finally, also in 2026, in related work, Gabrić et al. [10] described how to efficiently decode (i.e. find the location of any given n-tuple) certain classes of universal sequences for t-subsets and t-multisets.

Of particular relevance to the work here are the results of Sawada et al. [28]. They show ([28, Theorem 1]) that the (well known and very efficient) greedy

and FKM algorithms create the lexicographically smallest universal cycle for any subgraph whose set of edges is in $\mathcal{C}(n,k)$ — see Section 1.3. As we observed in Remarks 3.2, 4.1 and 5.2, any set containing all n-tuples of weight at least a specified lower bound is in $\mathcal{C}(n,k)$ for all three of the weight functions considered here. This immediately yields efficient methods of construction for these classes of universal sequence. While this by no means solves the problem of efficiently constructing universal sequences for all the cases we address here, it is at least a partial result. We leave the problem of finding such efficient methods of construction for all the cases considered in this paper for future work.

It would also be of interest to consider whether similar results could be obtained for other possible permutation-agnostic weight functions. Result 1.1 may well be relevant for other possible weight functions.

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