Orientable sequences with near optimal period

Sequences in London 2026, 5–6 February 2026

Chris Mitchell and Peter Wild

5th February 2026

1. Introduction: What are orientable sequences?

- A k-ary de Bruijn sequence of order n is an infinite periodic sequence of elements from $\{0, 1, \ldots, k-1\}$ in which every possible k-ary n-tuple occurs exactly once in a period.
- The period must be k^n, and there are many known methods of construction.
- Earliest known reference to constructing (and enumerating) such sequences is due to Sainte-Marie (1894), but better known work is by de Bruijn (1946) and Good (1947).
- Examples for $k = 2$ are: [0011] ($n = 2$), and [00010111] ($n = 3$).
- There are many applications, for example in stream ciphers, position location, and genome sequencing.
- De Bruijn sequences are examples of n-window sequences, periodic sequences in which any n-tuple occurs *at most once* in a period.

Orientable sequences

- An orientable sequence of order n (an $\mathcal{OS}_k(n)$) is a k-ary n-window sequence with the added property that an n-tuple occurs at most once in a period of a sequence *or its reverse*.
- First introduced in 1992, they have potential application in certain position location applications.
- For the binary case, a simple example for $n = 5$ has period 6 — a single period is [001011].
- The sequence and its reverse contain twelve distinct 5-tuples: 00101, 00110, 01001, 01011, 01100, 01101, and the complements of these 5-tuples.
- Examples for $k = 3$ are: [012] ($n = 2$) and [001201122] ($n = 3$).

The de Bruijn digraph

- The de Bruijn digraph is a key tool for analysing and constructing both de Bruijn and orientable sequences.
- This graph, otherwise known as the de Bruijn-Good graph, $B_k(n)$ is a directed graph with vertex set $\{0, 1, \dots, k-1\}^n$.
- An edge connects $(a_0, a_1, \dots, a_{n-1})$ to $(b_0, b_1, \dots, b_{n-1})$ iff $a_{i+1} = b_i$ for every i $(0 \leq i \leq n-2)$.
- It is simple to see that $B_k(n)$ is Eulerian, i.e. it is connected and every vertex has in-degree equal to its out-degree.
- If we identify an edge from $(a_0, a_1, \dots, a_{n-1})$ to $(b_0, b_1, \dots, b_{n-1})$ with the $(n+1)$-tuple $(a_0, a_1, \dots, a_{n-1}, b_{n-1})$, then a de Bruijn sequence of order $n+1$ corresponds to an Eulerian circuit in $B_k(n)$ — which must exist given $B_k(n)$ is Eulerian.
- There are, of course, efficient algorithms for finding such circuits.

The Lempel Homomorphism

- The Lempel D-function, originally defined only for $k = 2$, maps $B_2(n)$ to $B_2(n-1)$.
- D maps any binary n-tuple $(a_0, a_1, \ldots, a_{n-1})$ to $(a_1 - a_0, a_2 - a_1, \ldots, a_{n-1} - a_{n-2})$.
- D is a graph homomorphism from $B_2(n)$ to $B_2(n-1)$.
- Can extend definition to k-ary case, where D maps the k-ary n-tuple $(a_0, a_1, \ldots, a_{n-1})$ to $(a_1 - a_0, a_2 - a_1, \ldots, a_{n-1} - a_{n-2})$, where computations take place modulo k.
- The inverse of D has been widely used, e.g. to recursively construct de Bruijn sequences, observing that D^{-1} maps a circuit in $B_k(n-1)$ to a set of k circuits in $B_k(n)$.

Upper bounds on the period of orientable sequences

- Since any n-tuple can only occur once in a period in either direction, and symmetric n-tuples cannot occur, a trivial bound on the period of an $\mathcal{OS}_k(n)$ is
$$\frac{k^n - k^{\lfloor (n+1)/2 \rfloor}}{2}.$$
- However, apart from when $n = 2$ and k is odd, this bound is not sharp.
- The binary case is different from $k > 2$ — in particular, constant $(n-1)$-tuples and $(n-2)$-tuples cannot occur in a binary sequence, whereas they can for $k > 2$, so an $\mathcal{OS}_2(n)$ cannot exist for $n < 5$.
- Dai, Martin, Robshaw & Wild (1993) gave a bound for the binary case which is significantly sharper than the trivial bound.
- A bound for the $k > 2$ case which is a little sharper than the trivial bound was established a couple of years ago (Alhakim et al., 2024).

2. New upper bounds on the period

- In recent work (M and Wild, 2025b) we have established new upper bounds on the period of a k-ary orientable sequence (for $k > 2$), sharper than the 2024 bound.
- These bounds all derive from simple observations regarding the subgraph of the de Bruijn graph defined by the edges of an orientable sequence.
- If S is a k-ary orientable sequence of order n — an $\mathcal{OS}_k(n)$ — then we define B_S to be the subgraph of $B_k(n-1)$ with edges corresponding to the n-tuples appearing in either S or S^R (where S^R is the reverse of S).
- The n-tuples appearing in either S or S^R are, of course, all distinct since S is orientable.
- Since S and S^R define edge-disjoint (but not vertex-disjoint) Eulerian circuits in B_S, it follows that B_S must be Eulerian.
- This simple observation leads to the improved bounds, given we can identify cases where certain edges cannot occur in B_S.

Bounds — new and (old) — on the period of an $\mathcal{OS}_k(n)$

n	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
2	3	4	10	12	21	24
	(3)	(4)	(10)	(12)	(21)	(24)
3	9	20	50	84	147	216
	(9)	(22)	(50)	(87)	(147)	(220)
4	30	112	280	612	1134	1984
	(33)	(118)	(290)	(627)	(1155)	(2012)
5	99	452	1450	3684	8085	15896
	(105)	(478)	(1490)	(3777)	(8211)	(16124)
6	315	1958	7550	23019	58065	130332
	(336)	(2014)	(7680)	(23217)	(58464)	(130812)
7	972	7844	38100	138144	408072	1042712
	(1032)	(8062)	(38640)	(139317)	(410256)	(1046524)
8	3096	32390	193800	837879	2876496	8382492
	(3189)	(32638)	(194630)	(839157)	(2879835)	(8386556)
9	9423	129572	971350	5027304	20149437	67059992
	(9645)	(130558)	(974390)	(5034957)	(20166027)	(67092476)

3. Methods of construction

- As described in (Alhakim et al., 2024), can use the inverse Lempel homomorphism to go from an $\mathcal{OS}_k(n)$ of period m to an $\mathcal{OS}_k(n+1)$ of period km.
- However, it is non-trivial to ensure that D^{-1} yields a single sequence of period km rather than a set of $(n+1)$-tuple-disjoint sequences with periods summing to km.
- Moreover, some variants of the (inverse) Lempel homomorphism only yield 'negative' orientable sequences, in which the collection of all n-tuples and reverse negative n-tuples in a period are all distinct.
- Various approaches have been devised to fix this in recent work (Gabrić & Sawada, 2025) and (M & Wild, 2025a). Gabrić & Sawada showed how to join the multiple cycles produced, and in (M & Wild, 2025a) we constructed 'starter sequences' with special properties enabling repeated use of the Lempel homomorphism.
- The Gabrić & Sawada sequences have asymptotically maximal period.

A different approach: Antisymmetric subgraphs of the de Bruijn digraph

- This approach is described in (M & Wild, 2025b).
- A subgraph T of the de Bruijn digraph $B_k(n)$ is said to be *antisymmetric* if the following property holds.
- Suppose $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$ and $\mathbf{y} = (y_0, y_1, \ldots, y_{n-1})$ are k-ary n-tuples, i.e. vertices in $B_k(n)$.
- Then if $(\mathbf{x}, \mathbf{y})$ is an edge in T, then $(\mathbf{y}^R, \mathbf{x}^R)$ is *not* an edge in T.

From subgraph to sequence

- If S is an $\mathcal{OS}_k(n)$ of period m, then B_S is an antisymmetric Eulerian subgraph of $B_k(n-1)$ containing m edges.
- Antisymmetry follows from the definition of orientable.
- **More importantly**, if T is an antisymmetric Eulerian subgraph of $B_k(n-1)$ with m edges, then there exists an $\mathcal{OS}_k(n)$ S of period m with edge set T.
- Why? Since T is Eulerian there exists an Eulerian circuit. This Eulerian circuit corresponds to an n-window sequence, which is orientable since T is antisymmetric.

Antinegasymmetry

- A subgraph T of the de Bruijn digraph $B_k(n-1)$ is said to be *antinegasymmetric* if the following property holds.
- Suppose $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$ and $\mathbf{y} = (y_0, y_1, \ldots, y_{n-1})$ are k-ary n-tuples, i.e. vertices in $B_k(n)$.
- Then if $(\mathbf{x}, \mathbf{y})$ is an edge in T, then $(-\mathbf{y}^R, -\mathbf{x}^R)$ is *not* an edge in T.

From antinegasymmetry to antisymmetry

- If T is an antinegasymmetric subgraph of the de Bruijn digraph $B_k(n-1)$ with edge set E, then $D^{-1}(E)$, of cardinality $k|E|$, is the set of edges for an antisymmetric subgraph of $B_k(n)$, which, abusing our notation slightly, we refer to as $D^{-1}(T)$.
- If every vertex of T has in-degree equal to its out-degree, then the same applies to $D^{-1}(T)$.
- If T is connected and its edge set contains the all-one tuple, then $D^{-1}(T)$ is connected, i.e. in this case if T is Eulerian then so is $D^{-1}(T)$.

Constructing antinegasymmetric subgraphs

- If $0 \leq u \leq k-1$, set

$$f(u) = \begin{cases} u & \text{if } u \neq 0 \\ k/2 & \text{if } u = 0 \end{cases}$$

- If $\mathbf{u} = (u_0, u_1, \ldots, u_{n-1})$ is a k-ary n-tuple then the *pseudoweight* of $\mathbf{u}$ is defined to be the sum

$$w^*(\mathbf{u}) = \sum_{i=0}^{n-1} f(u_i)$$

  where the sum is computed in $\mathbb{Q}$.

- If E is the set of all k-ary n-tuples with pseudoweight less than $nk/2$, then E is the set of edges for an antinegasymmetric Eulerian subgraph of the de Bruijn digraph $B_k(n-1)$.

From antinegasymmetric to antisymmetric

- If, as in the previous slide, E is the set of all k-ary n-tuples with pseudoweight less than $nk/2$, then E contains the all-one n-tuple.
- Hence $D^{-1}(E)$ is a antisymmetric Eulerian subgraph of $B^k(n)$.
- This approach yields orientable sequences with largest possible period for $n = 3$ (all k) and $n = 4$ (k odd).

An example

Suppose $k = 3$ and $n = 3$. The ten 3-ary 3-tuples having pseudoweight less than 4.5 are listed below — this is E.

111		
011	101	110
001	010	100
112	121	211

$D^{-1}(E)$ consists of the 30 4-tuples given below.

0120	1201	2012						
0012	1120	2201	0112	1220	2001	0122	1200	2011
0001	1112	2220	0011	1122	2200	0111	1222	2000
0121	1202	2010	0101	1212	2020	0201	1012	2120

An $\mathcal{OS}_5(3)$ of period 30 containing these 4-tuples is:

[01201 21202 01012 22011 20011 12200]

.

Adding more n-tuples

- In recent (as yet unpublished) work, Peter Wild and I have looked at adding more n-tuples to E while preserving antinegasymmetry.
- Idea originates from observation that excluding all n-tuples with pseudoweight exactly $kn/2$ (the 'middle' value) is unnecessary.
- The set of n-tuples with pseudoweight exactly $kn/2$ can be divided into edge-disjoint circuits with period dividing n.
- If $(a_0, a_1, \ldots, a_{n-1})$ is a k-ary n-tuple, i.e. an edge in $B_k(n-1)$, then let $[a_0, a_1, \ldots, a_{n-1}]$ be the circuit in $B_k(n-1)$ consisting of edges

$$(a_0, a_1, \ldots, a_{n-1}), (a_1, a_2 \ldots, a_{n-1}, a_0), \ldots, (a_{n-1}, a_0, a_1, \ldots, a_{n-2}).$$

- The plan is to join the edges in some of these circuits to E while preserving antinegasymmetry (it is easy to show the result is Eulerian).

Negasymmetric circuits

- One of these circuits is said to be *negasymmetric* if there are edges, **a** and **b** say (not necessarily distinct), in the circuit such that $\mathbf{a} = -\mathbf{b}^R$.
- A negasymmetric circuit may contain a negasymmetric n-tuple, but it may not.
- The non-negasymmetric circuits come in complementary pairs. We can add one of each of these pairs to E, while preserving antinegasymmetry and the Eulerian property.
- Using the inverse Lempel Homomorphism, this yields a larger antisymmetric set of $(n+1)$-tuples — which is Eulerian — and hence orientable sequences with greater period.

Enumeration issues

- It is important to know how many of these complementary pairs there are, to know the period of the sequences obtained.
- We do this by enumerating negasymmetric circuits made up of n-tuples with pseudoweight $nk/2$ — in fact we enumerate negasymmetric circuits containing 0, 1 or 2 negasymmetric n-tuples, having shown these are the only possibilities.
- Assuming these all have the maximum period, this gives a lower bound on the number of edges we can add to E (which is sharp if n is prime).
- We can show that the orientable sequences generated this way have asymptotically optimal period (as both $k \to \infty$ and $n \to \infty$), and the periods are in practice larger than those in (Gabrić & Sawada, 2025).

An example

- Suppose $n = 5$ and $k = 3$. There are 51 3-ary 5-tuples with pseudoweight exactly $nk/2 = 7.5$.
- The 51 edges corresponding to these 5-tuples can be partitioned into 11 circuits, all but the first of which have period 5:

$$[00000], [00012], [00021], [00102], [00201], 01122], [01212],$$
$$[02211], [02121], [01221], [02112].$$

- The first nine of these circuits are negasymmetric and contain one negasymmetric 3-tuple, but the last two, i.e. [01221] and [02112], are not negasymmetric.
- The edges from one of these two circuits can be added to give an antinegasymmetric set of $96 + 5 = 101$ 5-tuples (there are 96 5-tuples with pseudoweight less than 7.5).
- The set of 101 5-tuples can be used to construct an antisymmetric set of $k \times 101 = 303$ 6-tuples, yielding an $\mathcal{OS}_3(6)$ of period 303 (greater than the previous record of 288 and close to the best known upper bound of 315).

4. Open questions

- A year ago at Sequences 2025, the only cases where the largest period was known was for $n = 2$ and $n = 3$ (odd k only).
- The new bounds and new construction methods mean we have now resolved the maximum period question for $n = 3$ (all k) and $n = 4$ (odd k).
- However, apart these small values of n, there remains a gap between the period of the longest known $\mathcal{OS}_k(n)$ and the best upper bound.
- This suggests further research is needed on two main problems:
  - tightening the upper bounds;
  - constructing sequences with periods closer to the upper bounds;

  so that (ideally) there is no gap.
- Eliminating the gap altogether seems difficult.

Largest known periods for $k = 2$

Order (n)	Maximum known period	Dai et al. bound
5	**6**	6
6	**16**	17
7	**36**	40
8	92	96
9	174	206
10	416	443

- Figures in bold represent maximal lengths as verified by search.
- For further details see the excellent website maintained by Joe Sawada: `http://debruijnsequence.org/db/orientable`

Largest known periods for $k > 2$ — as of Sequences 2025

Table: Largest known periods for $\mathcal{OS}_k(n)$ (and bounds) as of 2025

n	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
2	**3**	**4**	**10**	**12**	**21**
	(3)	(4)	(10)	(12)	(21)
3	**9**	20	**50**	72	**147**
	(9)	(22)	(50)	(87)	(147)
4	30	84	275	522	1127
	(33)	(118)	(290)	(627)	(1155)
5	90	368	1385	3360	7756
	(105)	(478)	(1490)	(3777)	(8211)
6	285	1608	7155	21150	56049
	(336)	(2014)	(7680)	(23217)	(58464)
7	879	7308	36890	135450	403389
	(1032)	(8062)	(38640)	(139317)	(410256)
8	2688	30300	187980	821940	2844408
	(3189)	(32638)	(194630)	(839157)	(2879835)

- Upper bound values are given in brackets.
- Figures in bold represent maximal lengths.

Largest known periods for $k > 2$ — latest results

Table: Largest known periods for an $\mathcal{OS}_k(n)$ (and bounds)

n	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
2	**3**	**4**	**10**	**12**	**21**	**24**
	(3)	(4)	(10)	(12)	(21)	(24)
3	**9**	**20**	**50**	**84**	**147**	**216**
	(9)	(20)	(50)	(84)	(147)	(216)
4	**30**	88	**280**	552	**1134**	1872
	(30)	(112)	(280)	(612)	(1134)	(1984)
5	93	404	1420	3546	8022	15640
	(99)	(452)	(1450)	(3684)	(8085)	(15896)
6	303	1744	7510	22272	57981	128544
	(315)	(1958)	(7550)	(23019)	(58065)	(130332)
7	954	7480	37980	136848	407736	1039568
	(972)	(7844)	(38100)	(138144)	(408072)	(1042712)
8	3006	31000	193140	830772	2874018	8359984
	(3096)	(32390)	(193800)	(837879)	(2876496)	(8382492)

- Upper bound values are given in brackets.
- Figures in bold represent maximal lengths.

5. Recent literature

- (Mitchell & Wild, 2022): IEEE Trans on Inf Thy **68** (2022) 4782–4789.
- (Alhakim et al., 2024): Cryptogr Commun **16** (2024) 1309–1326.
- (Gabrić & Sawada, 2025): Des Codes Cryptogr **93(7)** (2025) 2349–2367.
- (Mitchell & Wild, 2025a): Discret Appl Math **377** (2025) 242–259.
- (Mitchell & Wild, 2025b): arXiv.2507.02526.

Other resources

- Joe Sawada's page: `http://debruijnsequence.org/db/orientable`
- The Combinatorial Object Server: `http://combos.org/`