ON DIVISIONS AND DECOMPOSITIONS OF 1-DESIGNS

BY

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Thesis submitted to the University of London for the degree of Doctor of Philosophy, 1979.

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ABSTRACT of "On Divisions and Decompositions of 1-designs"

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A Point Division of a 1-(v,k,r) structure S is a partition of the points of S into classes such that the number of blocks through two points depends only on the classes to which they belong. This generalises the notion of a Group Divisible (GD) design and a number of results are obtained for Point Divisions of 1-designs which have well-known results for GD designs as corollaries.

Point Divisions are also closely linked to tactical divisions; in fact a tactical division of a 1-design is a tactical decomposition whose point classes form a certain special type of Point Division. Using this fact we obtain simple proofs of certain results on designs admitting tactical divisions. We also examine 2-designs whose duals admit Point Divisions, and show that this is equivalent to considering 2-designs having intersection number $k-r+\lambda$.

Using results obtained for Point Divisions of 1-designs, we go on to establish new results on GD designs, in particular we derive information about the duals of GD designs, and the properties of GD designs having certain special dual properties. We also obtain necessary and sufficient conditions for a symmetric GD design to have a GD dual.

Finally we give a general recursive method of construction for 1-structures admitting Point Divisions having constant class size. This method is used to construct both GD and 2-designs, and we use it to obtain two infinite families of strongly divisible 2-designs. One of these infinite families consists of quasi-residual designs, and we show that they are in fact residual designs. This establishes the existence of an infinite family of symmetric 2-designs.

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ACKNOWLEDGEMENTS

I am indebted to my supervisor Professor F.C. Piper and to Dr. H.J. Beker for their invaluable guidance and encouragement.

I would like to thank the Science Research Council for the postgraduate grant which supported me during my course of study, and I would also like to thank Chris Turner for typing this thesis.

CHAPTER 1 - PRELIMINARY DEFINITIONS AND RESULTS

Unless otherwise stated, all results may be found in Dembowski, [19].

1.1 Basic Definitions

An Incidence Structure (or a structure) is an ordered triple $(\underline{P},\underline{B},I)$, where $\underline{P}$ and $\underline{B}$ are finite non-empty sets whose elements are called points and blocks respectively; (we will use upper case latin letters to denote points and lower case latin letters to denote blocks). $I \subseteq \underline{P} \times \underline{B}$, and if $(P,x) \in I$ then the point P is said to be incident with the block x. It will often be convenient to associate a block with the set of points with which it is incident; and so we will frequently write $P \in x$ or $P\ I\ x$, and say P is on x or x contains P. We will also almost invariably use v and b to denote the number of points and blocks respectively.

A structure $\underline{S}$ is said to be uniform if there exists some constant k $(0<k<v)$ such that every block is incident with precisely k points. A t-structure, $\underline{S}$ $(t\geq 0)$, is a uniform structure having a constant λ such that every set of t points is incident with precisely λ common blocks. We will then say that $\underline{S}$ is a t-(v,k,λ) structure. Note that a 0-structure is just a uniform structure.

Result 1.1.1 If $\underline{S}$ is a t-(v,k,λ) structure then, for every s satisfying $0\leq s\leq t$, $\underline{S}$ is an s-(v,k,λ_s) structure, where $\lambda_s = \lambda\binom{v-s}{t-s} / \binom{k-s}{t-s}$.

In any t-structure we have $b = \lambda_0$, and if $t \geq 1$, we set $r = \lambda_1$. Result 1.1.1 immediately gives :-

Corollary 1.1.2   Suppose $\underline{S}$ is a t-(v,k,λ) structure. Then

(i) If $t \geq 1$ then $bk = vr$;

and (ii) if $t \geq 2$ then $\lambda_2(v-1) = r(k-1)$.

Clearly Result 1.1.1 indicates the non-existence of t-(v,k,λ) structures for most choices of v,k and λ ; since each $\lambda_s(0 \leq s \leq t)$ must be an integer.

We now define a class of structures which play a central role in this thesis.

A design is a uniform structure satisfying :-

(i) No two distinct blocks are incident with the same set of points; and

(ii) No two distinct points are incident with the same set of blocks.

As stated above, we will often identify a block with the set of points on it and, by (i), in a design the point set uniquely defines the block.

A t-design is a t-structure which is also a design. For a 1-design it is not difficult to show that $r = 1$ if and only if $k = 1$, and such a design is of little interest. So we will often assume that $r,k>1$. Also note that for a 2-(v,k,λ) design, axiom (ii) of the definition of design is unnecessary. For if two distinct points are incident with the same set of blocks, we immediately have $r = \lambda$, and hence (by Corollary 1.1.2(ii)) $v = k$, which contradicts the definition of uniform.

A 1-(v,k,r) design is said to be trivial if every subset of k points is a block. Clearly a trivial design is a t-design for every $t \leq k$.

The Connection Number of two points of a structure is the number of blocks incident with them both, and, dually, the Intersection Number of two blocks is the number of points incident with them both. If x,y are two blocks we will often write $|x \cap y|$ for the intersection number of x and y; (again considering x,y as point sets).

Finally we define a notion of connectedness. In a structure $\underline{S} = (\underline{P},\underline{B},I)$, a chain between two elements X,Y ($X,Y \in \underline{P} \cup \underline{B}$) is an ordered tuple $(X_0,X_1,X_2,\ldots,X_n)$ of elements of $\underline{P} \cup \underline{B}$ such that : $X = X_0$, $Y = X_n$ and $X_{i-1} I X_i$ for every $i (1 \le i \le n)$. Two elements of $\underline{P} \cup \underline{B}$ are connected if there exists a chain between them, and a structure $\underline{S}$ is connected if every pair of elements of $\underline{P} \cup \underline{B}$ are connected.

1.2 Incidence Matrices and Graphs

An incidence matrix $A = (a_{ij})$ for a structure $\underline{S}$ is a $v \times b$ matrix with its rows indexed by the points of $\underline{S}$ and its columns indexed by the blocks of $\underline{S}$, such that $a_{ij} = 1$ if the point corresponding to the i^{th} row is incident with the block corresponding to the j^{th} column, and $a_{ij} = 0$ otherwise. If we have some partition $\underline{P}_1,\ldots,\underline{P}_d$ of the points (or blocks) of $\underline{S}$, then an incidence matrix associated with this partition is arranged so that the first $|\underline{P}_1|$ rows (or columns) of A correspond to the points (blocks) of $\underline{P}_1$, the next $|\underline{P}_2|$ rows (columns) of A correspond to the points (blocks) of $\underline{P}_2$, and so on.

If $\underline{S}$ and $\underline{U}$ are two structures, then an isomorphism α from $\underline{S}$ onto $\underline{U}$ is a 1-1 mapping from the points of $\underline{S}$ onto the points of $\underline{U}$ and from the blocks of $\underline{S}$ onto the blocks of $\underline{U}$, such that PIx if and only if $P^{\alpha} I x^{\alpha}$. If there exists an isomorphism from $\underline{S}$ onto $\underline{U}$ then $\underline{S}$ and $\underline{U}$ are isomorphic and we write $\underline{S} \cong \underline{U}$.

Result 1.2.1 If $\underline{S}$, $\underline{U}$ are structures, and A,B are incidence matrices for $\underline{S}$ and $\underline{U}$ respectively, then $\underline{S} \cong \underline{U}$ if and only if there exist permutation matrices P,Q with $A = PBQ$.

Thus if $\underline{S}$ is a structure, all incidence matrices for $\underline{S}$ are equivalent, and hence have the same rank. So we define the <u>rank</u> of a structure $\underline{S}$ to be the rank of one (and hence all) of its incidence matrices.

We now state certain elementary facts about a structure in terms of its incidence matrix.

<u>Result 1.2.2</u> If A is the incidence matrix of a structure $\underline{S}$, then :-

(i) $\underline{S}$ is uniform if and only if $\underline{j}A = k\underline{J}$;

(ii) $\underline{S}$ is a 1-(v,k,r) structure if and only if $A\underline{j}^T = r\underline{j}^T$ and $\underline{j}A = k\underline{j}$; and

(iii) $\underline{S}$ is a 2-(v,k,λ) structure if and only if $\underline{j}A = k\underline{j}$ and $AA^T = (r-\lambda)I + \lambda J$;

where J is the all +1 matrix, I is the identity matrix, and $\underline{j}$ is the all +1 (row) vector (of appropriate sizes).

Note also that the off-diagonal entries of AA^T and A^TA are the connection and intersection numbers of the structure $\underline{S}$. These matrices are referred to as the <u>connection</u> and <u>intersection matrices</u> respectively.

Much information may also be derived from the eigenvalues and eigenvectors of the matrices AA^T and A^TA.

<u>Result 1.2.3 (Shrikhande and Bhagwandas, [45])</u> If $\underline{S}$ is a 1-(v,k,r) structure with incidence matrix A, then:-

(i) rk is an eigenvalue of AA^T.

(ii) rk is a simple eigenvalue of AA^T if and only if $\underline{S}$ is connected.

The proof of this result uses the following elementary results about real matrices : if $\underline{x}$ is an eigenvector for AA^T with eigenvalue $\phi \neq 0$, then $\underline{x}A$ is an eigenvector for A^TA with eigenvalue ϕ; AA^T and A^TA have the same non-zero eigenvalues with the same multiplicities.

We also obtain :-

<u>Lemma 1.2.4</u> If A is an incidence matrix of a 1-(v,k,r) structure $\underline{S}$ (with $r,k>1$), then $AA^T|\underline{j}^{\perp}$ has just one eigenvalue ϕ say, if and only if $\underline{S}$ is a 2-(v,k,λ) structure; in this case $\phi = r-\lambda$.

<u>Proof</u> If $\underline{S}$ is a 2-structure then $AA^T = (r-\lambda)I + \lambda J$, and so AA^T has eigenvectors $\underline{j}$, $\underline{e}_u$ $(1 \leq u \leq v-1)$ with eigenvalues rk and $r-\lambda$; where $\underline{e}_u = (\overbrace{0 --- 0}^{\leftarrow u-1 \rightarrow}\ 1\ \ -1\ \overbrace{0\ ---\ 0}^{\leftarrow v-u-1 \rightarrow})$. So $r-\lambda$ is the only eigenvalue of $AA^T|\underline{j}^{\perp}$.

If ϕ is the only eigenvalue of $AA^T|\underline{j}^{\perp}$, then any vector orthogonal to $\underline{j}$ must be an eigenvector of AA^T with eigenvalue ϕ. Hence, in particular, $\underline{e}_u AA^T = \phi \underline{e}_u$ for every u $(1 \leq u \leq v-1)$. Hence all off-diagonal entries are the same (and non-zero since $r,k>1$). I.e. $\underline{S}$ is a 2-structure and $\phi = r-\lambda$. ¤

<u>Remark</u> This Lemma is essentially the same as Corollary 1.2 of Kageyama and Tsuji, [30], although it is stated here in a slightly different form.

So the eigenvalues of a 2-structure $\underline{S}$ are rk and $r-\lambda$ with multiplicities 1 and $v-1$, and hence, since $r>\lambda$ by 1.1.2 (ii), $\underline{S}$ has v non-zero eigenvalues.

Immediately we have :-

<u>Result 1.2.5 (Fisher's Inequality)</u> If $\underline{S}$ is a 2-(v,k,λ) structure with incidence matrix A, then :-

$$v = \text{rank } AA^T = \text{rank } A = \text{rank } \underline{S} \leq b.$$

If $\underline{S}$ is a 2-(v,k,λ) structure with $v=b$, then $\underline{S}$ is said to be a symmetric 2-structure.

Finally we show a relationship between graphs and certain structures. A graph G (here a graph means an undirected graph without loops or multiple edges) is regular if every vertex is adjacent to a constant number of vertices. This number is called the valency of G. If $P_1,\ldots,P_v$ is some labelling of the vertices of a graph G (on v vertices), then the adjacency matrix $T = (t_{ij})$ of G corresponding to this labelling is a $v\times v$ matrix with $t_{ij} = 1$ if P_i and P_j are adjacent and $t_{ij} = 0$ otherwise; ($t_{ii} = 0$ for every i, $1\leq i\leq v$). For the results on graphs quoted below, see for instance, Cameron and van Lint, [17].

Result 1.2.6 If G is a graph with adjacency matrix T, then G is regular if and only if $\underline{j}T = \theta_0\underline{j}$, where θ_0 is the valency of G.

A graph G is complete if every pair of vertices are adjacent and G is null if it has no edges at all. The complement C(G) of a graph G is the graph with the same vertex set as G, such that two vertices are adjacent in C(G) if and only if they are not adjacent in G.

A regular graph G (G not null or complete) is said to be strongly regular if and only if for every pair of vertices P,P'; the number of vertices Q adjacent to both P and P' is a constant depending only on whether P and P' are adjacent or not. We shall refer to the eigenvalues of a graph G, meaning the eigenvalues of one (and hence all) of its adjacency matrices.

<u>Result 1.2.7</u> A graph G is strongly regular if and only if G is regular and $T|\underline{j}^{\perp}$ has two eigenvalues; these eigenvalues will be denoted by θ_1, θ_2 throughout.

The disjoint union of c complete graphs on m vertices each is a strongly regular graph denoted by $\Gamma(c,m)$. The complement of $\Gamma(c,m)$ is the "complete c-partite graph".

<u>Result 1.2.8</u> A strongly regular graph with eigenvalues θ_0 (the valency), θ_1 and θ_2 $(\theta_1 > \theta_2)$ is a $\Gamma(c,m)$ if and only if $\theta_1 = \theta_0$.

<u>Result 1.2.9</u> If G is a strongly regular graph on v vertices with eigenvalues θ_0 (the valency), θ_1 and θ_2; then C(G) has eigenvalues $v-1-\theta_0$ (the valency), $-1-\theta_1$ and $-1-\theta_2$ with the same multiplicities.

Suppose $\underline{S}$ is a 1-(v,k,r) structure with two connection numbers λ_1 and λ_2, say. Then we may form the graph $G(\underline{P},\lambda_i)$, called the <u>point graph</u> (with respect to λ_i), with vertices the points of $\underline{S}$, and with two vertices adjacent if and only if they have connection number λ_i. Then it is easy to see that $G(\underline{P},\lambda_1)$ and $G(\underline{P},\lambda_2)$ are complementary graphs. If A is an incidence matrix for $\underline{S}$, then we have :-

<u>Lemma 1.2.10</u> If $\underline{S}$ is a 1-(v,k,r) structure with two connection numbers λ_1 and λ_2 then :-

(i) $AA^T = (r-\lambda_j)I + \lambda_j J + (\lambda_i - \lambda_j)T$, where T is an adjacency matrix for $G(\underline{P},\lambda_i)$.

(ii) $G(\underline{P},\lambda_i)$ is regular with valency $((v-1)\lambda_j - r(k-1))/(\lambda_j - \lambda_i)$.

<u>Proof</u> (i) is straightforward, and, since $\underline{j}AA^T = rk\underline{j}$ (by Result 1.2.2 (ii)) we have $\underline{j}(\lambda_i-\lambda_j)T = (rk-(r-\lambda_j)-v\lambda_j)\underline{j}$, and (ii) follows. ¤

Finally, a similar argument shows that if a 1-structure $\underline{S}$ has two intersection numbers ρ_1,ρ_2; then it is possible to define the <u>block graph</u> of $\underline{S}$ (with respect to ρ_i), $G(\underline{B},\rho_i)$, and this graph is regular as above.

We also note that if a 1-structure $\underline{S}$ has two connection numbers then $G(\underline{P},\lambda_i)$ is strongly regular (i=1,2) if and only if $\underline{S}$ is a <u>Partially Balanced Design with two associate classes</u> (for more details see Section 1.4).

<u>1.3 Derived and Related Structures</u>

Given any structure $\underline{S} = (\underline{P},\underline{B},I)$ there are many ways to construct other, related structures from $\underline{S}$. We list some of these below.

(A) The <u>dual</u> of $\underline{S}$, which we denote by $\underline{S}^*$, is the structure $(\underline{B},\underline{P},I^*)$ where $(x,P)\in I^*$ if and only if $(P,x)\in I$; i.e. the roles of points and blocks are interchanged.

<u>Result 1.3.1</u> If $\underline{S}$ is a structure then :-

(i) $\underline{S}$ is a 1-(v,k,r) structure if and only if $\underline{S}^*$ is a 1-(b,r,k) structure.

(ii) $\underline{S}$ is a 1-(v,k,r) design if and only if $\underline{S}^*$ is a 1-(b,r,k) design.

Clearly $(\underline{S}^*)^* = \underline{S}$, and if A is any incidence matrix for $\underline{S}$, then A^T is an incidence matrix for $\underline{S}^*$.

Hence, by Lemma 1.2.4, if $\underline{S}$ is a symmetric 2-(v,k,λ) structure, then $\underline{S}^*$ is a 2-(b,r,k-r+λ) structure. But b=v and hence r=k (by Corollary 1.1.2 (ii)) and so $\underline{S}^*$ is a 2-(v,k,λ) structure. Finally $\lambda<r = k$, and so $\underline{S}$ and $\underline{S}^*$ are 2-designs and using Result 1.2.5 we have :

<u>Result 1.3.2</u> If $\underline{D}$ is a 2-(v,k,λ) structure then v=b if and only if $\underline{D}^*$ is a 2-structure; and in this case both $\underline{D}$ and $\underline{D}^*$ are 2-(v,k,λ) designs.

(B) The complement of $\underline{S}$, denoted by $C(\underline{S})$ is the structure $(\underline{P},\underline{B},\underline{P}\times\underline{B}-I)$; i.e. a point is incident with a block in $C(\underline{S})$ if and only if they are not incident in $\underline{S}$.

Clearly $C(C(\underline{S})) = \underline{S}$, and if A is any incidence matrix of $\underline{S}$, then $J-A$ is an incidence matrix of $C(\underline{S})$.

Using Result 1.2.2 we immediately have :-

Result 1.3.3 If $\underline{S}$ is a structure then :-

(i) $\underline{S}$ is uniform with k points on every block, if and only if $C(\underline{S})$ is uniform with $v-k$ points on every block;

(ii) $\underline{S}$ is a $1-(v,k,r)$ structure if and only if $C(\underline{S})$is a $1-(v,v-k,b-r)$ structure;

(iii) $\underline{S}$ is a $2-(v,k,\lambda)$ structure if and only if $C(\underline{S})$ is a $2-(v,v-k,b-2r+\lambda)$ structure (given $2\leq k\leq v-2$); and

(iv) $\underline{S}$ is a design if and only if $C(\underline{S})$ is a design.

Note that if $\underline{S}$ is a uniform structure with $k > \frac{v}{2}$ then $C(S)$ has $k < \frac{v}{2}$, and so we will sometimes assume that $k \leq \frac{v}{2}$ for uniform structures.

(C) Suppose that P is some point of $\underline{S}$. The Internal Contraction $\underline{S}_P$ of $\underline{S}$ is the structure $(\underline{P}-\{P\},\underline{B}',I')$ where $\underline{B}'$ is the set of blocks of $\underline{B}$ which are incident with $\underline{P}$, and I' is defined so that $(P,x)\in I'$ if and only if $(P,x)\in I$. Similarly we define the External Contraction $\underline{S}^P$ of $\underline{S}$ to be the structure $(\underline{P}-\{P\},\underline{B}'',I'')$ where $\underline{B}''$ is the set of blocks of $\underline{B}$ which are not incident with $P(\underline{B}''=\underline{B}-\underline{B}')$ and I'' is defined as for I'.

In an analagous way we define the internal and external structures $\underline{S}_x$ and $\underline{S}^x$ where x is some block of $\underline{S}$. So $\underline{S}_x$ consists of the points incident with x and all the blocks of $\underline{S}$ other than x (with the "same" incidence as $\underline{S}$), and $\underline{S}^x$ has as point set the points of $\underline{S}$ not incident with x, and all the blocks of $\underline{S}$ less x (again with incidence as in $\underline{S}$).

We may now state :

<u>Result 1.3.4</u> If $\underline{S}$ is a uniform structure with $k \geq 2$, then :-

(i) If $\underline{S}$ is a t-(v,k,λ) structure ($t \geq 1$), then $\underline{S}_p$ is a $(t-1)$-$(v-1,k-1,\lambda)$ structure.

(ii) If $\underline{S}$ contains no two blocks incident with the same point set, then neither does $\underline{S}_p$.

(iii) If $\underline{D}$ is a t-(v,k,λ) design ($t \geq 3$), then $\underline{D}_p$ is a $(t-1)$-$(v-1,k-1,\lambda)$ design.

<u>Result 1.3.5</u> If $\underline{S}$ is a uniform structure with $k \leq v-2$, then:-

(i) If $\underline{S}$ is a t-(v,k,λ) structure ($t \geq 1$), then $\underline{S}^p$ is a $(t-1)$-$(v-1,k,\lambda_{t-1}-\lambda)$ structure.

(ii) If $\underline{S}$ contains no two blocks incident with the same point set, then neither does $\underline{S}^p$.

(iii) If $\underline{D}$ is a t-(v,k,λ) design ($t=2, k \leq \frac{v}{2}$ or $t \leq 3$) then $\underline{D}^p$ is a $(t-1)$ - $(v-1,k,\lambda_{t-1}-\lambda)$ design.

The structures $\underline{S}_x$ and $\underline{S}^x$ do not necessarily have such pleasant properties; in fact they need not be 1-structures even if $\underline{S}$ is a 2-design. If $\underline{S}^x$ is a design it is known as a <u>residual</u> design of $\underline{S}$, and if a design $\underline{D}$ is isomorphic to a residual design of a structure $\underline{S}$, then $\underline{D}$ is said to be <u>embeddable</u> in $\underline{S}$.

In fact, for any uniform structure $\underline{S}$,

$\underline{S}^x$ is uniform if and only if every block intersects x in exactly the same number of points. If this occurs, and $\underline{S}$ is a 2-structure, then every two blocks of $\underline{S}$ intersect in the same number of points, and so, if a design $\underline{D}$ is embeddable in a

2-structure $\underline{S}$, $\underline{S}$ must be a symmetric 2-design (using Result 1.3.2). We may now state :

Result 1.3.6 If $\underline{D}$ is a symmetric 2-(v,k,λ) design ($k \leq \min\{\frac{v}{2}, v-2\}$), then $\underline{D}^x$ is a 2-$(v-k,k-\lambda,\lambda)$ design for any block x of $\underline{D}$.

Remark If $v > \frac{k}{2}$, $\underline{D}^x$ will still be a 2-$(v-k,k-\lambda,\lambda)$ structure, although it may contain repeated blocks (i.e. two blocks incident with precisely the same set of points).

It is now possible to consider which designs may be embeddable in symmetric 2-designs; and we have :-

Result 1.3.7 If a 2-(v,k,λ) design $\underline{D}$ is embeddable in a symmetric 2-design then $k-r+\lambda=0$, or , equivalently, $b-v = r-1$.

If a 2-design satisfies $b-v = r-1$ it is called a Quasi-residual Design, and indeed for $\lambda = 1$ or 2 every quasi-residual 2-(v,k,λ) design is a residual design of a 2-$(v+k+\lambda, k+\lambda,\lambda)$ design; see Hall and Connor, [21]. This result is not true for $\lambda \geq 3$, see, for example, the design of Bhattacharya, [10].

A considerable amount of work has been done on establishing sufficient conditions for certain quasi-residual designs to be residual, see for instance, Beker and Haemers, [7]; Bose, Shrikhande and Singhi, [16] and Singhi and Shrikhande, [47].

1.4 Group Divisible 1-designs

A considerable proportion of this thesis will be devoted to consideration of group divisible designs and certain related structures. In this section we give some of the basic results on these designs; we will assume $r,k>1$ throughout.

A <u>group divisible</u> 1-design (or just a <u>GD design</u>) is a 1-(v,k,r) design admitting a partition of the points into d classes $\underline{P}_1,\ldots,\underline{P}_d$ $(d<v)$ such that given two points P,Q $(P\in\underline{P}_i,$ $Q\in\underline{P}_j)$ the connection number of P and Q is a constant depending only on whether $i=j$ or not. The connection number of two points of the same point class and the connection number of points of different classes will be denoted by λ and λ' respectively; we assume throughout that $\lambda\neq\lambda'$. It can be shown that in a group divisible design there exists an $\ell>1$ so that $|\underline{P}_i|=\ell$ for every i.

If $\underline{D}$ is a GD design, and A is an incidence matrix for $\underline{D}$ associated with the point partition of $\underline{D}$, then :-

$$AA^T = \begin{array}{c} \\ \ell \\ \\ \ell \\ \\ \end{array} \begin{array}{c} \overset{\longleftarrow\ \ell\ \longrightarrow\ \longleftarrow\ \ell\ \longrightarrow}{} \\ \left(\begin{array}{c|c|c} \begin{matrix} r & & \lambda \\ & r\ddots & \\ \lambda & & r \end{matrix} & \lambda' & \cdots\lambda'\cdots \\ \hline \lambda' & \begin{matrix} r & & \lambda \\ & r\ddots & \\ \lambda & & r \end{matrix} & \cdots\lambda'\cdots \\ \hline \vdots\ \lambda'\ \vdots & \vdots\ \lambda'\ \vdots & \begin{matrix} r & & \\ & r\ddots & \\ & & r \end{matrix} \end{array}\right) \end{array}$$

So, by inspection, if $\underline{e}_u = (\overbrace{0\ 0\ \dots\ 0}^{u-1}\ 1\ -1\ \overbrace{0\ 0\ \dots\ 0}^{v-u-1})$

and $\underline{f}_w = (\overbrace{0\ 0\ \dots 0}^{(w-1)\ell}\ \overbrace{1\ 1\ \dots\ 1}^{\ell}\ \overbrace{-1\ -1\ \dots -1}^{\ell}\ \overbrace{0\ 0\ \dots\ 0}^{(v-w-1)\ell})$; then :-

$$\underline{j}(AA^T) = ((v-\ell)\lambda' + (\ell-1)\lambda + r)\underline{j}$$

$$\underline{e}_u(AA^T) = (r-\lambda)\underline{e}_u \quad (1\leq u\leq v-1;\ u\neq t\ell \text{ for any } t)$$

$$\underline{f}_w(AA^T) = (r+(\ell-1)\lambda-\ell\lambda')\underline{f}_w \qquad (1\leq w\leq \ell-1).$$

But $\underline{j}A = k\underline{j}$ and $\underline{j}A^T = r\underline{j}$ (Result 1.2.2.(ii)), and so $\underline{j}(AA^T) = rk\underline{j}$. Hence :-

<u>Result 1.4.1</u> If $\underline{D}$ and A are as above, then :-

(i) $v = d\ell$;

(ii) $(v-\ell)\lambda' + (\ell-1)\lambda = r(k-1)$;

(iii) The eigenvalues of AA^T are : $(r-\lambda)$, $(rk-v\lambda')$ and rk with multiplicities $v-\ell$, $\ell-1$ and 1 respectively; and

(iv) $|AA^T| = rk(r-\lambda)^{v-\ell}(rk-v\lambda')^{\ell-1}$.

Since $r,k>1$, Result 1.2.3(ii) gives :-

<u>Result 1.4.2</u> If $\underline{D}$ is GD then $\underline{D}$ is connected if and only if $\lambda'>0$.

So we will normally assume $\lambda'>0$ for GD designs, or otherwise $\underline{D}$ consists of the disjoint union of d 2-(ℓ,k,λ) designs.

Again from Result 1.4.1, since AA^T is positive semi-definite, we have :-

<u>Result 1.4.3 (Bose and Connor, [14])</u> If $\underline{D}$ is GD then $rk\geq v\lambda'$.

By definition of design, $r>\lambda$, and so if $\underline{D}$ is GD, then $rk>v\lambda'$ if and only if AA^T is non-singular. This led Bose and Connor (in [14]) to classify GD designs as follows :-

Result 1.4.4 If $\underline{D}$ is GD then either :-

(i) $rk = v\lambda'$ and $b \geq \text{rank } A = \text{rank } AA^T = v-d+1$ (in which case $\underline{D}$ is said to be Semi-regular GD, or SRGD), or

(ii) $rk > v\lambda'$ and $b \geq \text{rank } A = \text{rank } AA^T = v$ (in this case $\underline{D}$ is said to be Regular GD, or RGD).

In fact Bose and Connor's definition of GD designs allows two points to be incident with the same set of blocks, and hence allows the case $r=\lambda$. Such designs they call Singular GD (SGD) designs. The definition used here does not permit SGD designs, and in any case it can be shown that an SGD design consists of a 2-design with each point repeated ℓ times.

SRGD designs may be characterised by a point class - block intersection property as follows :-

Result 1.4.5 (Bose and Connor, [14]) If $\underline{D}$ is GD then $\underline{D}$ is SRGD if and only if every block is incident with precisely $\frac{k}{d}$ points of each point class.

The following is also known for SRGD designs :-

Result 1.4.6 (Connor, [18]) If $\underline{D}$ is SRGD then $\lambda < \lambda'$.

We now consider the situation of equality in Result 1.4.4.

Result 1.4.7 (Roy and Laha, [36]; Saraf, [37])

If $\underline{D}$ is SRGD then $b=v-d+1$ if and only if $\underline{D}^*$ is a 2-design. In this case $\underline{D}^*$ is a $2\text{-}(b,r,k-r+\lambda)$ design.

Result 1.4.8 (Connor, [18]) If $\underline{D}$ is RGD, $b=v$ and $(rk-v\lambda', \lambda-\lambda') = 1$, then $\underline{D}^*$ is RGD with the same parameters as $\underline{D}$.

RGD designs with $b=v$ whose duals are not RGD with the same parameters as $\underline{D}$ seem rare. Any GD design satisfying $b=v$ is called symmetric and necessary and sufficient conditions for a symmetric GD design to have a GD dual will be obtained below.

For an example of a symmetric SRGD design whose dual is not GD, see Connor, [18].

Group Divisible Designs may be regarded as a special class of Partially Balanced Designs with two associate classes. To define these objects we first need the following concept.

An <u>m-class Association Scheme</u> on a set X ($|X|=v$) is a partition $\underline{A} = A_1, A_2, \ldots, A_m$ ($A_i \neq \phi$ for any i) of the set $\underline{P}_2(X)$ of all two-element subsets of X, having the following property. If $\{x,y\} \in A_h$, then the number of $z \in X$ with $\{x,z\} \in A_i$ and $\{y,z\} \in A_j$ is a constant p_{ij}^h independent of the choice of x and y. The sets $A_1, \ldots, A_m$ are called the <u>classes</u> of the association scheme, and if $\{x,y\} \in A_i$, then x and y are said to be <u>i^{th} associates.</u>

We immediately have $p_{ij}^h = p_{ji}^h$ for every i,j,h ($1 \leq i,j,h \leq m$). The unique partitions of $\underline{P}_2(X)$ with m=1 and $m=v(v-1)/2$ are association schemes on X, but since they are uninteresting we shall usually assume that $1<m<v(v-1)/2$.

<u>Result 1.4.9</u> Let $\underline{A}$ be an m-class association scheme on X. Given $x \in X$, then the number of i^{th} associates of X depends only on i and is independent of the choice of x. We denote this number by n_i and we have :-

(i) $n_i = \sum_{j=1}^{m} p_{ij}^h + \delta_{ih}$ for every i,h($1 \leq i,h \leq m$);

(ii) $n_h p_{ij}^h = n_i p_{jh}^i = n_j p_{ih}^j$ for every i,j,h($1 \leq i,j,h \leq m$);

and (iii) $\sum_{i=1}^{m} n_i = v-1$; where δ_{ih} is the Kronecker Delta.

A <u>Partially Balanced Design with m associate classes</u> (a <u>PBD(m)</u>) is a 1-design $\underline{D}$ together with an m-class association scheme $\underline{A}=A_1, A_2, \ldots, A_m$ defined on $\underline{P}$ (the point set of $\underline{D}$) such that there exist constants $\lambda_1,\lambda_2,\ldots,\lambda_m$ (not necessarily distinct) with the property that given $\{P,Q\}\in A_i$, then the connection number of P and Q is λ_i.

A 2-design is clearly a PBD(1). Also if G is a graph with vertex set X, and $A_1 = \{\{x,y\}\subseteq X \mid x \text{ adjacent to } y \text{ in } G\}$; $A_2 = \underline{P}_2(X) - A_1$, then G is strongly regular if and only if $\underline{A} = \{A_1,A_2\}$ is a 2-class association scheme on X. If G is a strongly regular graph then the association scheme obtained in this way will be referred to as the association scheme corresponding to G, and vice versa. Furthermore, it is not difficult to see that if $\underline{D}$ is a PBD(2) with association scheme corresponding to a strongly regular graph G, then $G \cong G(\underline{P},\lambda_1)$.

So a GD design is a PBD(2) with respect to the association scheme corresponding to $\Gamma(d,\ell)$, with $\lambda_1=\lambda$, $\lambda_2=\lambda'$ and $G(\underline{P},\lambda) \cong \Gamma(d,\ell)$.

For a general PBD(m) we also have :-

<u>Result 1.4.10</u> The parameters of a PBD(m) satisfy

$$\sum_{i=1}^{m} \lambda_i n_i = r(k-1).$$

We shall mainly be concerned with PBD(2)'s and so we now give the following useful Lemmas :-

<u>Lemma 1.4.11</u> A 1-design $\underline{D}$ with incidence matrix A and precisely two connection numbers, is a PBD(2) if and only if $AA^T|\underline{j}^\perp$ has two eigenvalues.

<u>Proof</u> Since $\underline{D}$ has two connection numbers, λ_1,λ_2, say, then by Lemma 1.2.10 :

$AA^T = (r-\lambda_2)I + \lambda_2 J + (\lambda_1-\lambda_2)T$ where T is an adjacency matrix for $G(\underline{P},\lambda_1)$. By Results 1.2.6 and 1.2.7, $G(\underline{P},\lambda_1)$ is strongly regular if and only if $T|\underline{j}^\perp$ has two eigenvalues, and $\underline{j}$ is an eigenvalue of T.

But $\underline{j}$ is always an eigenvalue of T, since $\underline{j}$ is an eigenvalue of AA^T, I and J (by Result 1.2.2 (ii)). So by the above remarks, $\underline{D}$ is a PBD(2) if and only if $G(\underline{P},\lambda_1)$ is strongly regular, i.e. if and only if $T|\underline{j}^{\perp}$ has two eigenvalues, and using the above equation connecting AA^T and T the result follows. ⌘

<u>Lemma 1.4.12</u> Suppose that $\underline{D}$ is a PBD(2), and T (the adjacency matrix of the graph $G(\underline{P},\lambda_1)$) is such that $T|\underline{j}^{\perp}$ has eigenvalues $\theta_1,\theta_2(\theta_1>\theta_2)$ and $\underline{j}T = \theta_0\underline{j}$. Then $\underline{D}$ is GD with connection numbers $\lambda = \lambda_1$, $\lambda' = \lambda_2$ if and only if $\theta_0 = \theta_1 = ((v-1)\lambda_2 - r(k-1))/(\lambda_2-\lambda_1)$, $\theta_2 = -1$, multiplicity $(\theta_0=\theta_1) = d$ and multiplicity $(\theta_2) = v-d$.

<u>Proof</u> By Result 1.2.8 $G(\underline{P},\lambda_1) \cong \Gamma(d,\ell)$ if and only if $\theta_0=\theta_1$. But, by the remarks above, $G(\underline{P},\lambda_1) \cong \Gamma(d,\ell)$ if and only if $\underline{D}$ is GD with $\lambda = \lambda_1$, $\lambda' = \lambda_2$.

If $\underline{D}$ is GD with $\lambda=\lambda_1$, $\lambda' = \lambda_2$, then, by Result 1.4.1, AA^T has eigenvalues $rk, rk-v\lambda_2$ and $r-\lambda_1$ with multiplicities 1, $d-1$ and $v-d$. By Lemma 1.2.10 :-

$AA^T = (r-\lambda_2)I + \lambda_2 J + (\lambda_1-\lambda_2)T$ where T is an adjacency matrix for $G(\underline{P},\lambda_1)$. So

$T = [(r-\lambda_2)I + \lambda_2 J - AA^T]/(\lambda_2-\lambda_1)$ and T has eigenvalues : $\theta_0 = \theta_1 = ((v-1)\lambda_2 - r(k-1))/(\lambda_2-\lambda_1)$ and $\theta_2 = ((r-\lambda_2)-(r-\lambda_1))/(\lambda_2-\lambda_1) = -1$ with multiplicities d and $v-d$ respectively; and the Lemma follows. ⌘

<u>1.5 Tactical Decompositions and Resolutions</u>

Tactical Decompositions are closely linked to certain generalisations of group divisible 1-designs.

A <u>Tactical Decomposition</u> $T(\underline{S})$ of an incidence structure $\underline{S}$ is a partition of the points and blocks of $\underline{S}$ into classes

$\underline{P}_1,\ldots,\underline{P}_d$ and $\underline{B}_1,\ldots,\underline{B}_c$ respectively, such that :-

(i) The number of points of $\underline{P}_i$ incident with a block of $\underline{B}_j$ depends only on i and j and is denoted by β_{ij};

and (ii) The number of blocks of $\underline{B}_j$ incident with a point of $\underline{P}_i$ is a constant γ_{ij} depending only on the choice of classes.

Clearly every structure admits the <u>trivial</u> tactical decomposition whose point and block classes consist of the singleton point and block sets. We assume from now on that every tactical decomposition is non-trivial.

A <u>Tactical Division</u>, $T(\underline{S})$, of a 1-structure $\underline{S}$ is a Tactical Decomposition whose point classes $\underline{P}_1,\ldots,\underline{P}_d$ satisfy :-

(i) Given two distinct points P,Q ($P\in\underline{P}_i$, $Q\in\underline{P}_j$), the connection number of P and Q depends only on the choices of i and j and is denoted by λ_{ij};

and (ii) There exists a λ such that $\lambda_{ii} = \lambda$ for every i $(1\leq i\leq d)$.

For a 2-structure, the terms tactical division and tactical decomposition are equivalent.

We may now state :-

<u>Result 1.5.1 (Beker, [6])</u> If $T(\underline{D})$ is a tactical division of a 1-design $\underline{D}$, then $b+d \geq v+c$.

Result 1.5.1 is a generalisation of a result of Block, [11]. Tactical Divisions satisfying $b+d = v+c$ are of special interest, and are called <u>strong</u>.

Result 1.5.2 (Beker, [6]) If $T(\underline{D})$ is a tactical division of a 1-design $\underline{D}$, then the following are equivalent:

(i) $T(\underline{D})$ is strong;

(ii) The intersection number of two distinct blocks depends only on their block classes;

and (iii) Every pair of distinct blocks from the same block class intersect in $k-r+\lambda$ points.

A Tactical Division with just one point class is called a resolution, and a design admitting a resolution will be called resolvable. By definition it is clear that a resolvable design must be a 2-design.

Result 1.5.3 (Hughes and Piper, [23]) If $R(\underline{D})$ is a resolution of a 2-design $\underline{D}$, then $b+1 \geq v+d$. $R(\underline{D})$ is strong (i.e. $b+1 = v+d$) if and only if every block class contains $m = \frac{b}{c}$ blocks, and the intersection number of two distinct blocks is $k-r+\lambda$ or $\frac{k^2}{v}$ depending only on whether the blocks are from the same or different block classes respectively.

The parameters of a strongly resolvable 2-design (i.e. a 2-design admitting a strong resolution) may be characterised as follows :-

Result 1.5.4 (Harris, [22]) If $\underline{D}$ is a strongly resolvable 2-(v,k,λ) design with m blocks in every block class, then : $v = \mu m^2/\sigma^2$, $k = \mu m/\sigma$ and $\lambda = (\mu m-\sigma)/(m-1)$ where $\mu = k^2/v$ and $\sigma = \gamma_{1j}$ for every j.

A parallelism of an incidence structure is a tactical decomposition with one point class, satisfying $\gamma_{ij} = 1$ for every $j (1 \leq j \leq c)$. Clearly a parallelism of a 2-design is a special type of resolution, and in this case, when this resolution is strong the 2-design is said to be affine.

As a Corollary to Results 1.5.3 and 1.5.4 we have :

Result 1.5.5. (Bose, [12]) If a 2-design $\underline{D}$ admits a parallelism with c block classes, then $b+1 \geq v+c$. $\underline{D}$ is affine (i.e $b+1 = v+c$) if and only if every block class contains $m = \frac{b}{c}$ blocks and the intersection number of two distinct blocks is 0 or $\frac{k^2}{v}$ depending only on whether the blocks are from the same or different block classes respectively. In this case $\underline{D}$ is a $2-(\mu m^2, \mu m, (\mu m-1)/(m-1))$ design, $(\sigma=1)$.

An affine plane is then simply an affine design with $\mu=1$.

The notation and definitions used here are by no means standard. Many authors (e.g. Bose, [12]; Kageyama, [28]; Shah, [39]; and Shrikhande, [40],[41],[43]) call a structure resolvable if it admits a parallelism.

The definition of strongly resolvable given above corresponds to the definition of affine α-resolvable of Shrikhande and Raghavarao, [44], in the case when $\underline{D}$ is a 2-design. In general a 1-design $\underline{D}$ is "affine α-resolvable" if and only if $\underline{D}^*$ is SRGD. Also, for a 2-design, the definition of resolvability given above is a more general notion than that of "α-resolvability"; (an α-resolution of a 1-design is a tactical decomposition with one point class such that $\gamma_{1j} = \alpha$ for every j). In fact, in the 2-design case it corresponds precisely with the definition of $(\mu_1,\mu_2,\ldots,\mu_t)$-resolvable of Kageyama, [28].