CHAPTER 3 - GROUP DIVISIBLE DESIGNS

In this Chapter we consider group divisible designs; in particular we study the properties of the duals of such designs.

Bose and Connor, [14] first introduced the concept of a GD design. One main reason for introducing these designs was for work in the design of experiments; in this context we often need structures with given parameters. Unfortunately 2-designs (or B.I.B.D's) with suitable parameters do not always exist, and GD designs are in some sense the "closest" to 2-designs.

Connor in [18] studied symmetric GD designs and gave bounds on the intersection numbers of these designs as well as giving a sufficient condition for a symmetric GD design to have a GD dual. Agrawal, [2], (among others), gave bounds on the intersection numbers of all GD designs and Shrikhande and Bhagwandas, [45], derived a further sufficient condition for a symmetric GD design to have a GD dual.

Apart from this work, it remains an open question as to which symmetric GD designs have a GD dual. The only symmetric GD design known to the author which does not have a GD dual is the SRGD design to be found in [18], and hence it seems possible that all symmetric RGD designs have a GD dual.

Symmetric GD designs with a GD dual have been studied by several authors. In particular, they have been studied in connection with Baer subdesigns of symmetric 2-designs by Bose and Shrikhande, [15] and Bose, [13]. The underlying theme of this Chapter is the study of such GD designs; firstly necessary and sufficient conditions are obtained for a GD design to have a GD dual, and secondly, properties of such designs are examined.

In Section 3.1 general properties of the dual of a GD design are considered; the known results on bounds on the size of intersection numbers of GD designs are listed and some new results of this type are obtained. At the end of this section a result of Connor's is generalised and some useful corollaries are obtained.

GD designs having only two intersection numbers are discussed in Section 3.2. Necessary and sufficient conditions are given for such a design to have as its dual a PBD(2), and for such designs further necessary and sufficient conditions are obtained for the dual to be group divisible. Lastly the properties of GD designs with GD duals are investigated.

In Section 3.3 we obtain two sets of necessary and sufficient conditions for a symmetric GD design to have a GD dual. Furthermore, two well known results giving sufficient conditions for a symmetric design to have a GD dual are derived using results obtained previously, and finally we derive two new results which give further sufficient conditions of this type.

We assume that $\underline{D}$ is a GD design throughout, and also that $\lambda' > 0$; (hence $\underline{D}$ is connected and $r, k > 1$).

3.1 Intersection Numbers of Group Divisible Designs

There are many results giving bounds on the intersection numbers of GD designs, and we now give a summary of such results. Note that some of these bounds have been established for larger classes of designs, of which GD designs are a special case; in particular Agrawal, [2], has obtained bounds on the intersection numbers of all 1-designs in terms of the eigenvalues of AA^T (where A is an incidence matrix of the design in question).

Trivially $0 \leq |x \cap y| \leq k-1$ for every pair of distinct blocks x,y of a design.

Result 3.1.1 (Agrawal, [2]) If x,y are two distinct blocks of $\underline{D}$, then :-

(i) If $\lambda<\lambda'$ then $k-r+\lambda \leq |x \cap y| \leq 2\lambda' v/b - 2\ell(\lambda'-\lambda)/b - (k-r+\lambda)$;

(ii) If $\lambda>\lambda'$ then $(k-r+\lambda) - \ell(\lambda-\lambda') \leq |x \cap y| \leq 2\lambda' v/b + \ell(\lambda-\lambda') - (k-r+\lambda)$.

Remarks The bounds of Result 3.1.1 were originally given in a slightly different form, but the bounds above can easily be obtained from those of [2] by application of Result 1.4.1(ii). Saraf, [37] and Shah, [38] have obtained similar results, but Agrawal in [2] and [3] has shown that the bounds above are in every case at least as good.

For the symmetric case (i.e. if v=b) we have:

Result 3.1.2 (John, [26]) If $\underline{D}$ is a symmetric SRGD design and x,y are two distinct blocks of $\underline{D}$, then :-

$\lambda \leq |x \cap y| \leq 2\lambda'-\lambda-1$.

Remarks John established his upper bound by showing that in the symmetric SRGD case, the upper bounds of Connor, [18]; Saraf, [37] and Agrawal, [2], all have integer part $2\lambda'-\lambda-1$, and hence are all equivalent. John went on to show that these bounds are "best possible" by constructing a family of SRGD designs whose intersection numbers attain both the bounds of Result 3.1.2.

Result 3.1.3 (Connor, [18]) If $\underline{D}$ is a symmetric RGD design and x,y are two distinct blocks of $\underline{D}$, then :-

(i) If $\lambda<\lambda'$ then $\lambda \leq |x\cap y| \leq \lambda'(r-\lambda)/(r^2-v\lambda')$;

(ii) If $\lambda>\lambda'$ then $\lambda'(r-\lambda)/(r^2-v\lambda') \leq |x\cap y| \leq \lambda$.

There also exist results similar to those of Majumdar, [32], (see Result 2.5.1), for certain GD designs.

Result 3.1.4 (Agrawal, [2]) Suppose x,y are two distinct blocks of $\underline{D}$. If $\lambda<\lambda'$ and $|x\cap y| = k-r+\lambda$ or if $\lambda>\lambda'$ and $|x\cap y| = (k-r+\lambda) - \ell(\lambda-\lambda')$ then $|x\cap z| = |y\cap z|$ for every block z ($z\neq x$ or y).

Another similar result is Result 2.2.7, part of which we restate here for completeness.

Result 3.1.5 (Saraf, [37]) If $\underline{D}$ is a SRGD design and x,y are two distinct blocks of $\underline{D}$, then $|x\cap y| = k-r+\lambda$ if and only if $|x\cap z| = |y\cap z|$ for every block z ($x\neq x$ or y).

Agrawal, [2] and Neumaier, [33] have obtained further results of this type for general 1-designs.

We now prove some results similar to those of Connor, [18]. The method of proof employed here is a generalisation of a proof (due to R.M. Wilson) of analagous results for 2-designs. We first establish :-

<u>Lemma 3.1.6</u> Suppose A is a $v\times b$ matrix with linearly independent rows. Then :-

(i) $P = A^T(AA^T)^{-1}A$ is the matrix of the orthogonal projection of $\mathbb{R}^b$ onto the subspace $\text{row}(A) = U$, say;

(ii) $Q = I - P$ is the matrix of the orthogonal projection of $\mathbb{R}^b$ onto $U^\perp$.

<u>Proof</u> (i) Suppose $\underline{x} \in \mathbb{R}^b$. Then we may (uniquely) express $\underline{x}$ in the form $\underline{x} = \underline{x}_1 + \underline{x}_2$ where $\underline{x}_1 \in U$ and $\underline{x}_2 \in U^\perp$. Hence $\underline{x}_1 = \underline{y}A$ for some $\underline{y} \in \mathbb{R}^v$ and $A\underline{x}_2{}^T = \underline{0}^T$.

Then $\underline{x}P = (\underline{x}_1+\underline{x}_2)P = \underline{x}_1A^T(AA^T)^{-1}A + \underline{x}_2A^T(AA^T)^{-1}A$

$$= \underline{y}AA^T(AA^T)^{-1}A + \underline{0}(AA^T)^{-1}A = \underline{y}A = \underline{x}_1.$$

(ii) Trivial. ¤

<u>Lemma 3.1.7</u> If Q is as in Lemma 3.1.6 and Q_1 is any principal submatrix of Q, then :-

(i) $|Q_1| \geq 0$;

(ii) If order $(Q_1) > b-v$ then $|Q_1| = 0$; (where the order of Q_1 is the size of Q_1).

<u>Proof</u> By Lemma 3.1.6, every vector in U is an eigenvector for Q of eigenvalue 0, and every vector in $U^\perp$ is an eigenvector for Q of eigenvalue 1. U has dimension v (since A has linearly independent rows) and hence $U^\perp$ has dimension $b-v$. So rank $Q = b-v$ and Q is positive semi-definite since all its eigenvalues are non-negative.

(i) If Q_1 is a principal submatrix of Q, then Q_1 is positive semi-definite since $\underline{x}Q_1\underline{x}^T = \underline{x}'Q\underline{x}'^T \geq 0$; where $\underline{x}'$ has the entries of $\underline{x}$ in the places corresponding to the rows and columns of Q_1, and zeros elsewhere. So $|Q_1| \geq 0$.

(ii) rank $Q_1 \leq$ rank $Q = b - v$. So if Q_1 has order greater than b-v, then $|Q_1| = 0$. ¤

We now use Lemma 3.1.7 to obtain two results of Connor's on RGD designs. We first require :-

<u>Lemma 3.1.8</u> If <u>D</u> is a RGD design and A is an incidence matrix for <u>D</u> associated with the group division, then:-

$$(AA^T)^{-1} = \frac{rk(rk-v\lambda')I - \lambda'(r-\lambda)J - rk(\lambda-\lambda')K}{rk(rk-v\lambda')(r-\lambda)}$$

where $K = I_d \otimes J_\ell$ and $\otimes$ indicates the Kronecker Product.

<u>Proof</u> As we observed in Section 1.4 above

$AA^T = \lambda' J + (\lambda-\lambda')K + (r-\lambda)I$. Hence the product of AA^T and the expression on the R.H.S. of the above equation =

$[rk(rk-v\lambda')(r-\lambda)I+\lambda'\{rk((rk-v\lambda')-\ell(\lambda-\lambda'))-(v\lambda'+(r-\lambda)-\ell(\lambda-\lambda'))$
$(r-\lambda)\}J+rk(\lambda-\lambda')((rk-v\lambda')-\ell(\lambda-\lambda')-(r-\lambda))K]/rk(rk-v\lambda')(r-\lambda) = I$
(applying Result 1.4.1(ii)). ¤

Theorem 3.1.9 and Corollary 3.1.10 below are due to Connor [18].

<u>Theorem 3.1.9</u> Suppose that <u>D</u> is regular, and that $Q = (q_{ij})$ is a b×b matrix, with

$$q_{uw} = [\lambda'(r-\lambda)+(\lambda-\lambda')\sum_{i=1}^{d} s^i_{uu}s^i_{ww}-(rk-v\lambda')\,s(u,w)]/(rk-v\lambda')(r-\lambda);$$

where s_{uw} is as defined in Lemma 2.2.4 and

$$s(u,w) = \begin{cases} k-r+\lambda & \text{if } u = w \\ s_{uw} & \text{if } u \neq w \end{cases}.$$

If Q_1 is any principal submatrix of Q, then :-

(i) $|Q_1| \geq 0$;

(ii) If order $(Q_1) >$ b-v then $|Q_1| = 0$.

<u>Proof</u> Suppose that A is an incidence matrix for $\underline{D}$ associated with the group division of $\underline{D}$. Then A is a $v\times b$ matrix with linearly independent rows (by Result 1.4.4) and so, by Lemmas 3.1.6 and 3.1.7 if Q_1 is any principal submatrix of $Q = I - A^T(AA^T)^{-1}A$ then Q_1 satisfies conditions (i) and (ii) of the theorem. Hence we need only show that $I - A^T(AA^T)^{-1}A$ has entries q_{uw} as above.

By Lemma 3.1.8, $A^T(AA^T)^{-1}A =$

$$[rk(rk-v\lambda')A^TA-\lambda'(r-\lambda)A^TJA-rk(\lambda-\lambda')A^TKA]/rk(rk-v\lambda')(r-\lambda).$$

By Result 1.2.2(ii), $A^TJA = rkJ$, and, employing the notation of Lemma 2.1.4,

$$KA = \begin{array}{c} \\ \updownarrow \ell \\ \\ \updownarrow \ell \\ \vdots \end{array} \overset{\xleftrightarrow{\quad b \quad}}{\begin{pmatrix} s^1_{11} & s^1_{22} & & s^1_{bb} \\ \vdots & \vdots & \cdots & \vdots \\ s^1_{11} & s^1_{22} & & s^1_{bb} \\ \hline s^2_{11} & s^2_{22} & & s^2_{bb} \\ \vdots & \vdots & \cdots & \vdots \\ s^2_{11} & s^2_{22} & & s^2_{bb} \\ \hline & & \vdots & \\ & & & \end{pmatrix}}$$

Hence A^TKA has $\sum_{i=1}^{d} s^i_{uu}s^i_{ww}$ in the u^{th} row and w^{th} column,

and hence $I-A^T(AA^T)^{-1}A$ has entry

$$[(rk-v\lambda')(r-\lambda)\delta_{uw} - (rk-v\lambda')s_{uw} + \lambda'(r-\lambda) + (\lambda-\lambda')\sum_{i=1}^{d} s^i_{uu}s^i_{ww}] / (rk-v\lambda')(r-\lambda)$$

in its u^{th} row and w^{th} column. The theorem then follows. ¤

<u>Corollary 3.1.10</u> If <u>D</u> is symmetric and regular then :-

(i) $\sum_{i=1}^{d}(s^i_{uu})^2 = (r^2-v\lambda') + \ell\lambda'$;

(ii) $s_{uw} = \lambda' + \left(\sum_{i=1}^{d} s^i_{uu}s^i_{ww} - \ell\lambda'\right)(\lambda-\lambda')/(r^2-v\lambda')$ for every

u,w $(1\leq u,w\leq b)$.

<u>Proof</u> (i) Let Q_1 be any principal submatrix of Q of order one, i.e. Q_1 has as its single entry :

$$[\lambda'(r-\lambda)+(\lambda-\lambda')\sum_{i=1}^{d}(s^i_{uu})^2 - (r^2-v\lambda')\lambda]/(r^2-v\lambda')(r-\lambda).$$

By Theorem 3.1.9(ii) this entry must be zero, and (i) follows after applying Result 1.4.1(ii).

(ii) Let Q_1 be any principal submatrix of Q of order two. By (i) above, Q_1 has diagonal entries zero, and its off diagonal entries are both

$$[\lambda'(r-\lambda)+(\lambda-\lambda')\sum_{i=1}^{d} s^i_{uu}s^i_{ww} - (r^2-v\lambda')s_{uw}]/(r^2-v\lambda')(r-\lambda).$$

Again, using Theorem 3.1.9(ii), these entries must be zero, and (using Result 1.4.1(ii)) (ii) follows. ¤

More generally we have :-

<u>Corollary 3.1.11</u> If <u>D</u> is regular then :-

$$\sum_{i=1}^{d}(s^i_{uu})^2 \geq (rk-v\lambda') + \ell\lambda' - (r-k)(rk-v\lambda')/(\lambda-\lambda').$$

<u>Proof</u> Let Q_1 be any principal submatrix of Q of order one. $|Q_1| \geq 0$ by Theorem 3.1.9(i) and since $(rk-v\lambda')(r-\lambda) > 0$ the Corollary follows (after applying Result 1.4.1(ii)). ¤

<u>Remark</u> Using Corollary 3.1.10 we may also derive Result 3.1.3.

Clearly $0 \leq \sum_{i=1}^{d}(s^i_{uu}-s^i_{ww})^2 = \sum_{i=1}^{d}(s^i_{uu})^2 + \sum_{i=1}^{d}(s^i_{ww})^2 - 2\sum_{i=1}^{d}s^i_{uu}s^i_{ww}$

$= 2\left((r^2-v\lambda')+\ell\lambda' - \sum_{i=1}^{d}s^i_{uu}s^i_{ww}\right)$, using Corollary 3.1.10(i).

Hence $\sum_{i=1}^{d}s^i_{uu}s^i_{ww} \leq (r^2-v\lambda')+\ell\lambda'$. Also $0 \leq s^i_{uu}$ for every i,

and so $0 \leq \sum_{i=1}^{d}s^i_{uu}s^i_{ww} \leq (r^2-v\lambda')+\ell\lambda'$. Result 3.1.3 follows

after applying Corollary 3.1.10(ii).

We now obtain some new results on intersection numbers of GD designs, similar in nature to Results 3.1.1 - 3.1.5 above. We first require :

<u>Lemma 3.1.12</u> Using the notation of Lemma 2.2.4 :

$$\sum_{\substack{t=1\\t\neq u\\t\neq w}}^{b}(s_{ut}-s_{wt})^2 = (\lambda-\lambda')\sum_{i=1}^{d}(s^i_{uu}-s^i_{ww})^2+2\left(s_{uw}-(k-r+\lambda)\right)(k-s_{uw}).$$

<u>Proof</u> Immediate from Lemma 2.2.5. ¤

We now have :

<u>Theorem 3.1.13</u> If x,y are two distinct blocks of $\underline{D}$, then :-

(i) If $\lambda<\lambda'$ then $k-r+\lambda \leq |x\cap y|$ with equality if and only if $|x\cap z| = |y\cap z|$ for every block $z (z\neq x$ or $y)$ and $|x\cap \underline{P}_i| = |y\cap \underline{P}_i|$ for every point class $\underline{P}_i$ $(1\leq i\leq d)$;

(ii) If $\lambda>\lambda'$ and $|x\cap z| = |y\cap z|$ for every block z, then $|x\cap y| \leq k-r+\lambda$;

(iii) Any two of the following imply the third :

(a) $|x\cap y| = k-r+\lambda$;

(b) $|x\cap y| = |y\cap z|$ for every block z ($z\neq x$ or y) ;

(c) $|x\cap \underline{P}_i| = |y\cap \underline{P}_i|$ for every class $\underline{P}_i$ $(1\leq i\leq d)$.

<u>Proof</u> The Theorem follows immediately from Lemma 3.1.12 since $k-s_{uw} \geq 0$. ¤

<u>Remarks</u> (i) gives the lower bound of Result 3.1.1(i) and a stronger result than Result 3.1.4(i). If $\underline{D}$ is SRGD then $\lambda<\lambda'$ and $|x\cap \underline{P}_i| = |y\cap \underline{P}_i|$ for every x,y and $\underline{P}_i$: so in this case (i) above just becomes Result 2.2.7.

<u>Theorem 3.1.14</u> If $\underline{D}$ is a symmetric RGD design and x,y are two distinct blocks of $\underline{D}$, then :-

(i) If $\lambda<\lambda'$ then $\lambda \leq |x\cap y| \leq \lambda+\ell(\lambda'-\lambda)$;

(ii) If $\lambda>\lambda'$ then $\lambda-\ell(\lambda-\lambda') \leq |x\cap y| \leq \lambda$;

(iii) $|x\cap z| = |y\cap z|$ for every block z ($z\neq x$ or y) if and only if $|x\cap y| = \lambda-\ell(\lambda-\lambda')$ or $|x\cap y| = \lambda$.

<u>Proof</u> Set $x = x_u$, $y = x_w$. In the symmetric case Lemma 3.1.12 gives :-

$$\sum_{\substack{t=1\\t\neq u\\t\neq w}}^{b}(s_{ut}-s_{wt})^2 = (\lambda-\lambda')\sum_{i=1}^{d}(s^i_{uu}-s^i_{ww})^2 + 2(s_{uw}-\lambda)(r-s_{uw}).$$

Using Corollary 3.1.10 we also have :

$$\sum_{i=1}^{d}(s^i_{uu}-s^i_{ww})^2 = 2\{\lambda(r-\lambda)+(\lambda-\lambda')\ell\lambda-(r^2-v\lambda')s_{uw}\}/(\lambda-\lambda').$$

Substituting in the above and manipulating we obtain :

$$\sum_{\substack{t=1 \\ t\neq u,w}}^{b} (s_{ut}-s_{wt})^2 = 2(s_{uw}-\lambda)\big((\lambda-\ell(\lambda-\lambda'))-s_{uw}\big).$$

The theorem then follows. ¤

<u>Remarks</u> The lower bounds of (i),(ii) are the same as the lower bounds of Result 3.1.1(i),(ii); also, the lower bound of (i) and the upper bound of (ii) are the same as the bounds of Result 3.1.3(i),(ii).

We now consider certain GD designs whose duals are divisible, and then obtain some further results on the intersection numbers of such designs. Bounds are obtained which are analagous to those obtained by Beker and Haemers, [7], for decompositions of 2-designs; (see also section 2.5 above). As we shall see below, GD designs whose duals are also GD are in fact strongly divisible, and in the regular case they are symmetric.

<u>Lemma 3.1.15</u> Suppose that $\underline{B}_1,\ldots,\underline{B}_c$ is a CLP Division of $\underline{D}^*$ with $\rho = k-r+\lambda$ and $m_j=m$ for every j. Then, for every i,j $(1\leq i\leq d, 1\leq j\leq c)$ there exists a constant β_{ij}, such that every block of $\underline{B}_j$ is incident with β_{ij} points of $\underline{P}_i$; and

$$m\sum_{\substack{j=1 \\ j\neq u,w}}^{c} (\rho_{uj}-\rho_{wj})^2 = (\lambda-\lambda')\sum_{i=1}^{d}(\beta_{iu}-\beta_{iw})^2+2m\big(\rho_{uw}-(k-r+\lambda)\big)$$

$$\big((r-\lambda)/m+(k-r+\lambda)-\rho_{uw}\big), \text{ for every } u,w\ (u\neq w).$$

Proof β_{ij} exists for every i,j by Theorem 3.1.13(iii).

By Lemma 2.1.5 :-

$$m\sum_{j=1}^{c}\rho_{ju}^{2} = \lambda'\sum_{i=1}^{d}\sum_{\substack{j=1\\ j\neq i}}^{d}\beta_{iu}\beta_{ju}+\lambda\sum_{i=1}^{d}\beta_{iu}^{2}+(r-\lambda)\sum_{i=1}^{d}\beta_{iu}-(k^2-\rho^2)$$

$$= \lambda'\sum_{i=1}^{d}\sum_{\substack{j=1\\ j\neq i}}^{d}\beta_{iu}\beta_{ju}+\lambda\sum_{i=1}^{d}\beta_{iu}^{2}+(r-\lambda)k-(k^2-\rho^2); \text{ and}$$

$$m\sum_{j=1}^{c}\rho_{ju}\rho_{jw} = \lambda'\sum_{i=1}^{d}\sum_{\substack{j=1\\ j\neq i}}^{d}\beta_{iu}\beta_{jw}+\lambda\sum_{i=1}^{d}\beta_{iu}\beta_{iw}+(r-\lambda)\sum_{i=1}^{d}s^{i}_{(ue)(wf)}-2(k-\rho)\rho_{uw}$$

$$= \lambda'\sum_{i=1}^{d}\sum_{\substack{j=1\\ j\neq i}}^{d}\beta_{iu}\beta_{jw}+\lambda\sum_{i=1}^{d}\beta_{iu}\beta_{iw}+(r-\lambda)\rho_{uw}-2(k-\rho)\rho_{uw}\ .$$

Hence $m\sum_{j=1}^{c}(\rho_{ju}-\rho_{jw})^2=(\lambda-\lambda')\sum_{i=1}^{d}(\beta_{iu}-\beta_{iw})^2+2(r-\lambda)\big(\rho_{uw}-(k-r+\lambda)\big)$

(using the fact that $p=k-r+\lambda$ and $\sum_{j=1}^{d}\beta_{ju}=k$).

So, finally, $m\sum_{\substack{j=1\\ j\neq u}}^{c}(\rho_{ju}-\rho_{jw})^2=(\lambda-\lambda')\sum_{i=1}^{d}(\beta_{iu}-\beta_{iw})^2+$

$$2m\big(\rho_{uw}-(k-r+\lambda)\big)\big((r-\lambda)/m+(k-r+\lambda)-\rho_{uw}\big).\quad ¤$$

We now have :

Theorem 3.1.16 If $\underline{D}$ satisfies the conditions of Lemma 3.1.15, and if $\underline{B}_u,\underline{B}_w$ are two distinct classes of the CLP Division of $\underline{D}^*$, then :-

(i) If $\lambda<\lambda'$ then $k-r+\lambda \le \rho_{uw} \le (r-\lambda)/m + (k-r+\lambda)$ with equality in either one of the inequalities if and only if $\rho_{uj} = \rho_{wj}$ for every j ($j \neq u$ or w) and $\beta_{iu} = \beta_{iw}$ for every i;

(ii) If $\lambda>\lambda'$ and $\rho_{uj} = \rho_{wj}$ for every j ($j \neq u$ or w), then $\rho_{uw} \le k-r+\lambda$ or $\rho_{uw} \ge (r-\lambda)/m + (k-r+\lambda)$;

(iii) Any two of the following imply the third :-

(a) $\rho_{uw} = k-r+\lambda$ or $\rho_{uw} = (r-\lambda)/m + (k-r+\lambda)$;

(b) $\rho_{uj} = \rho_{wj}$ for every j ($j \neq u$ or w);

(c) $\beta_{iu} = \beta_{iw}$ for every i.

<u>Proof</u> Immediate from Theorem 3.1.14 and Lemma 3.1.15. ¤

We next give a generalisation of a result of Connor, [18], on RGD designs.

<u>Theorem 3.1.17</u> If <u>D</u> is a symmetric GD design, then (with the notation of Lemma 2.2.4) :

$$\sum_{i=1}^{v} \sum_{j=1}^{v} (s_{ij}-\lambda)(s_{ij}-\lambda') = 0.$$

<u>Proof</u> Consider the trivial P Division of <u>D</u>* and apply Lemma 2.1.6(ii),(iii) to obtain :

$$\sum_{i=1}^{v} \sum_{\substack{j=1 \\ j\neq i}}^{v} s_{ij} = vk(k-1)$$

$$= v(\ell-1)\lambda + v(v-\ell)\lambda' \quad \text{(by Result 1.4.1(ii))},$$

and $$\sum_{i=1}^{v} \sum_{\substack{j=1 \\ j\neq i}}^{v} s_{ij}^2 = v(v-\ell)\lambda'^2 + v(\ell-1)\lambda^2.$$

Then $$\sum_{i=1}^{v} \sum_{\substack{j=1 \\ j\neq i}}^{v} (s_{ij}-\lambda)(s_{ij}-\lambda') = 0. \quad ¤$$

The following two corollaries lead to two well known results on symmetric GD designs; (see Section 3.3. below).

<u>Corollary 3.1.18</u> If $\underline{D}$ is a symmetric GD design and $|\lambda-\lambda'| = 1$, then $\underline{D}$ has precisely two intersection numbers : λ and λ'.

<u>Proof</u> Since λ,λ' are consecutive integers every term on the L.H.S. of the equation of Theorem 3.1.17 must be non-negative, and so $\underline{D}$ has at most two intersection numbers: λ and λ'. But both these values must occur or else $\underline{D}^*$ is a symmetric 2-design, and hence $\underline{D}$ is a 2-design. ¤

<u>Corollary 3.1.19</u> If $\underline{D}$ is a symmetric RGD design and $(r^2-v\lambda',\lambda-\lambda') = 1$, then $\underline{D}$ has precisely two intersection numbers : λ and λ'.

<u>Proof</u> By Corollary 3.1.10(ii), $s_{uw} = \lambda'+a(\lambda-\lambda')/(r^2-v\lambda')$ where $a \in \mathbb{Z}$. Since $(r^2-v\lambda',\lambda-\lambda') = 1$, $a/(r^2-v\lambda') \in \mathbb{Z}$ and so $s_{uw} = \lambda'+b(\lambda-\lambda')$; $b \in \mathbb{Z}$. So s_{uw} cannot lie between λ and λ' and so, as in the previous corollary $(s_{ij}-\lambda)(s_{ij}-\lambda') \geq 0$ for every i,j. The result follows immediately as in Corollary 3.1.18. ¤

Finally, we also have two further corollaries :

<u>Corollary 3.1.20</u> If $\underline{D}$ is a symmetric GD design then either :-

(i) $\underline{D}$ has precisely two intersection numbers : λ and λ';

or (ii) There exist blocks x,x',y,y' satisfying :

$$|x\cap y'| < \lambda' < |y\cap y'| \quad \text{and} \quad |x\cap x'| \neq \lambda \neq |y\cap y'|.$$

<u>Proof</u> Suppose $\lambda<\lambda'$. Then, if $|x\cap y| \leq \lambda'$ for every pair of blocks x,y, all the terms of $\sum_{i=1}^{v}\sum_{\substack{j=1\\j\neq i}}^{v}(s_{ij}-\lambda)(s_{ij}-\lambda')$ are non-

positive (by Result 3.1.1(ii)), and so λ,λ' are the only intersection numbers. If this is not the case then there exist blocks y,y' with $\lambda<\lambda'<|y\cap y'|$, and hence the sum above has a positive term. So, by Theorem 3.1.17 and Result 3.1.1(ii) there exist blocks x,x' with $\lambda<|x\cap x'|<\lambda'$.

A similar argument holds if $\lambda>\lambda'$, and the Corollary follows. ¤

Corollary 3.1.21 Suppose $\underline{D}$ is symmetric. Then if

(i) $\lambda<\lambda'$ and $d=2$, or

(ii) $\lambda>\lambda' = 1$, then $\underline{D}$ has precisely two intersection numbers : λ and λ'.

Proof (i) By Result 3.1.1(i), if x,y are two distinct blocks of $\underline{D}$ then $\lambda\leq|x\cap y|\leq 2\lambda'-(\lambda'-\lambda)-\lambda=\lambda'$. Hence all the terms of $\sum_{i=1}^{v}\sum_{\substack{j=1\\ j\neq i}}^{v}(s_{ij}-\lambda)(s_{ij}-\lambda')$ are non-positive, and so by Theorem 3.1.17, λ and λ' are the only intersection numbers of $\underline{D}$.

(ii) Since $\lambda>\lambda'$, $\underline{D}$ is regular by Result 1.4.6. Hence, by Result 3.1.3(ii), if x,y are two distinct blocks of $\underline{D}$, then:

$\lambda'(r-\lambda)/(r^2-v\lambda') \leq |x\cap y| \leq \lambda$. But $r-\lambda>0$, $r^2-v\lambda'>0$ and $\lambda'=1$, so we have

$\lambda' = 1 \leq |x\cap y| \leq \lambda$. As for (i), by Theorem 3.1.17, λ and λ' are the only intersection numbers. ¤

3.2 Group Divisible Designs with Two Intersection Numbers

In this section we examine GD designs with exactly two intersection numbers. Results obtained are analagous to known results for quasi-symmetric 2-designs ; for example compare Result 2.5.3 of Shrikhande and Bhagwandas, [45], Goethals and Seidel, [20] and Theorem 2.5.9 with Theorems 3.2.2 and 3.2.5 below.

We also show that if both $\underline{D}$ and $\underline{D}^*$ are GD, then $\underline{D}$ is strongly divisible, generalising a result of Bose, [13].

To study these designs it is necessary to consider the regular and semi-regular cases separately. We first suppose $\underline{D}$ is regular :

Lemma 3.2.1 If $\underline{D}$ is RGD with two intersection numbers μ_1, μ_2, and T_i is an adjacency matrix for $G(\underline{B}, \mu_i)$, then T_i has eigenvalues : $((k-\mu_j)-(rk-b\mu_j))/(\mu_j-\mu_i)$ (the valency); $((k-\mu_j)-(r-\lambda))/(\mu_j-\mu_i)$; $((k-\mu_j)-(rk-v\lambda'))/(\mu_j-\mu_i)$ and $(k-\mu_j)/(\mu_j-\mu_i)$ with multiplicities : 1, v-d, d-1 and b-v.

Proof By Lemma 1.2.10 (i), $T_i = ((k-\mu_j)I+\mu_j J-A^T A)/(\mu_j-\mu_i)$. By Lemma 1.2.10(ii) $\underline{j}T_i = [((k-\mu_j)-(rk-b\mu_j))/(\mu_j-\mu_i)]\underline{j}$, where A is an incidence matrix for $\underline{D}$. $\underline{D}$ is RGD and so $AA^T|\underline{j}^{\perp}$ has eigenvalues $r-\lambda$ and $rk-v\lambda'$ with multiplicities v-d and d-1; ($r-\lambda$, $rk-v\lambda'$ both non zero). So $A^TA|\underline{j}^{\perp}$ has eigenvalues $r-\lambda$, $rk-v\lambda'$ and 0 with multiplicities v-d,d-1 and b-v. Using the expression above for T_i in terms of A^TA the Lemma follows. ¤

<u>Theorem 3.2.2</u> If $\underline{D}$ is RGD with two intersection numbers ρ, ρ' ($\rho < \rho'$ if and only if $\lambda < \lambda'$); then :

(i) $\underline{D}^*$ is a PBD(2) if and only if $b=v$;

(ii) If $b=v$, then $\underline{D}^*$ is GD if and only if $\rho' = \lambda'$. In this case $\underline{D}^*$ is GD with the same parameters as $\underline{D}$; (i.e. $G(\underline{B},\rho) \cong \Gamma(d,\ell)$, $\rho=\lambda$ and $\rho'=\lambda'$).

<u>Proof</u> Let A be an incidence matrix for $\underline{D}$.

(i) $A^TA|\underline{j}^{\perp}$ has eigenvalues $r-\lambda$, $rk-v\lambda'$ and 0 with multiplicities $v-d$, $d-1$ and $b-v$. All these eigenvalues are distinct ($rk-v\lambda'>0, r-\lambda>0$ and $rk-v\lambda' \neq r-\lambda$ by Result 1.4.1.(ii)).
(i) then follows by Lemma 1.4.11.

(ii) Let T and T$'$ be adjacency matrices for $G(\underline{B},\rho)$ and $G(\underline{B},\rho')$ respectively. By Lemma 3.2.1 T has eigenvalues $\theta_0 = \big((k-\rho')-(k^2-v\rho')\big)/(\rho'-\rho)$;
$\theta_1 = \big((k-\rho')-(k^2-v\lambda')\big)/(\rho'-\rho)$ and
$\theta_2 = \big((k-\rho')-(k-\lambda)\big)/(\rho'-\rho)$ with multiplicities 1, $d-1$ and $v-d$; ($\theta_1>\theta_2$ since $rk-v\lambda'>r-\lambda$ if and only if $\lambda>\lambda'$ for an arbitrary GD design-immediate from Result 1.4.1 (ii)).

Also T$'$ has eigenvalues $\theta_0' = \big((k-\rho)-(k^2-v\rho)\big)/(\rho-\rho')$, $\theta_1' = \big((k-\rho)-(k-\lambda)\big)/(\rho-\rho')$ and $\theta_2' = \big((k-\rho)-(k^2-v\lambda')\big)/(\rho-\rho')$ with multiplicities $1, v-d$ and $d-1$; ($\theta_1'>\theta_2'$ as previously).

By Lemma 1.4.12, $\underline{D}^*$ is GD if and only $\theta_0=\theta_1$ or $\theta_0'=\theta_1'$.

But if $\theta_0'=\theta_1'$, $\underline{D}^*$ is GD with $v-d+1$ classes in the group division (considering the multiplicity of θ_1' and using Lemma 1.4.12). This cannot occur since $1<d|v$ and $(v-d+1)|b = v$. So $\underline{D}^*$ is GD if and only if $\theta_0=\theta_1$; i.e. if and only if $\rho'=\lambda'$. By Lemma 1.4.12, $\theta_2= -1$ and so $\lambda=\rho$. Finally, considering the multiplicity of θ_1, we see that the group division of $\underline{D}^*$ has d classes. ¤

Secondly we consider SRGD designs. We first need :

Lemma 3.2.3 If $\underline{D}$ is SRGD and $\underline{D}^*$ is GD, then $\underline{D}^*$ is SRGD with $\rho=k-r+\lambda$; also the group divisions form a strong tactical division of $\underline{D}$.

Proof By Result 3.1.5, $\rho=k-r+\lambda$. By Result 1.4.5 every block contains equally many points of each point class, and applying Theorem 3.1.13(iii) to $\underline{D}^*$ we see that the group divisions form a tactical division of $\underline{D}$ which is strong by Result 1.5.2. Finally $\underline{D}^*$ is SRGD by Theorem 2.4.5. ¤

Remark Kageyama in [29], Corollary 2.6, essentially proves the first part of the above Lemma.

Lemma 3.2.4 If $\underline{D}$ is SRGD with two intersection numbers μ_1,μ_2 and T_i is an adjacency matrix for $G(\underline{B},\mu_i)$, then T_i has eigenvalues : $((k-\mu_j)-(rk-b\mu_j))/(\mu_j-\mu_i)$ (the valency), $((k-\mu_j)-(r-\lambda))/(\mu_j-\mu_i)$ and $(k-\mu_j)/(\mu_j-\mu_i)$ with multiplicities $1, v-d$ and $b-v+d-1$.

Proof c.f. Lemma 3.2.1. ¤

<u>Theorem 3.2.5</u> If $\underline{D}$ is SRGD with two intersection numbers ρ, ρ' $(\rho<\rho')$ then :

(i) $\underline{D}^*$ is a PBD(2);

(ii) $\underline{D}^*$ is GD if and only if $\rho' = \lambda' v/b$. In this case $\rho = k-r+\lambda$ and the group division of $\underline{D}^*$ has $b-v+d$ classes.

<u>Proof</u> Let A be an incidence matrix for $\underline{D}$.

(i) Immediate from Lemma 1.4.11 since $A^TA|\underline{j}^{\perp}$ has eigenvalues $r-\lambda$ and 0 with multiplicities $v-d$ and $b-v+d-1$.

(ii) Let T be an adjacency matrix for G $(\underline{B},\rho)$. If $\underline{D}^*$ is GD then $\underline{D}^*$ is SRGD by Lemma 3.2.3. But, for any SRGD design $\lambda<\lambda'$ (Result 1.4.6) and so (since $\rho<\rho'$) $\underline{D}^*$ is GD if and only if G $(\underline{B},\rho) \cong \Gamma(c,m)$ for some c and m. Now by Lemma 3.2.4, T has eigenvalues $\theta_0 = \{(k-\rho')-(rk-b\rho')\}/(\rho'-\rho)$, $\theta_1 = (k-\rho')/(\rho'-\rho)$ and $\theta_2 = \{(k-\rho')-(r-\lambda)\}/(\rho'-\rho)$ with multiplicities $1, b-v+d-1, v-d$; $(\theta_1>\theta_2)$. So, by Lemma 1.4.12, $\underline{D}^*$ is GD if and only if $\theta_0 = \theta_1$, i.e. if and only if $\rho' = rk/b$ $=\lambda' v/b$ (since $rk=v\lambda'$). The rest of the theorem follows immediately from Lemma 1.4.12. ¤

<u>Remarks</u> Theorems 3.2.2(i) and 3.2.5(i) show that if $\underline{D}$ has two intersection numbers, then a necessary and sufficient condition for the block graphs to be strongly regular is that $\underline{D}$ is symmetric RGD or SRGD. That this is a necessary condition has previously been established by Shrikhande and Bhagwandas in [45]. In fact they give the more general result that if $\underline{D}$ is a symmetric or non-regular PBD(2) with two

intersection numbers then $\underline{D}^*$ is a PBD(2) with two intersection numbers then $\underline{D}^*$ is a PBD(2).

To conclude this section we now show :

<u>Theorem 3.2.6</u> If $\underline{D}^*$ is GD then the group divisions of $\underline{D}$ and $\underline{D}^*$ form a strong tactical division of $\underline{D}$, and either :-

(i) $\underline{D},\underline{D}^*$ are SRGD; $\rho=k-r+\lambda$, $\rho'=\lambda' v/b$, $\beta_{ij} = k/d$ and $\gamma_{ij} = r/c$ for every i,j; or

(ii) $b = v$; $\underline{D},\underline{D}^*$ are RGD with the same parameters and $\beta_{ij} = \gamma_{ij}$ for every i,j.

<u>Proof</u> By Lemma 3.2.3 $\underline{D}$ is SRGD if and only if $\underline{D}^*$ is SRGD. In this case (by this Lemma) $\rho=k-r+\lambda$, and the group divisions form a strong tactical division of $\underline{D}$. $\beta_{ij}=k/d$ and $\gamma_{ij}=r/c$ by Result 1.4.5, $\rho'=\lambda' v/b$ by Theorem 3.2.5(ii) and we have (i).

If $\underline{D},\underline{D}^*$ are RGD, then, by Theorem 3.2.2, $\underline{D}$ is symmetric and $\underline{D},\underline{D}^*$ have the same parameters. Finally Lemma 2.1.7 gives $\beta_{i(jt)}= \gamma_{(is)j}$ and hence the group divisions form a tactical division with $\beta_{ij} = \gamma_{ij}$, which is strong by Result 1.5.2. ¤

3.3 Symmetric Group Divisible Designs

The first theorem in this section considerably improves an earlier result of Bose who essentially showed (iii) ⇒ (i) of Theorem 3.3.1, which is in itself a special case of Theorem 3.2.6. Theorem 3.3.2 is a result similar in nature to Theorem 3.2.2 and 3.2.5 in that it studies GD designs with two intersection numbers, but it shows that a stronger result is possible in the symmetric case; note the strong resemblance between this theorem and an earlier result (Theorem 2.5.9) for quasi-symmetric designs.

Finally two well-known sets of sufficient conditions for a symmetric GD design to have a GD dual are derived; furthermore two new results of a similar nature are also given.

Theorem 3.3.1 If $\underline{D}$ is symmetric, then the following are equivalent :-

(i) $\underline{D}$ admits a strong tactical division with point classes the classes of the group division of $\underline{D}$;

(ii) $\underline{D}^*$ admits a CLP Division with d classes and $\rho=\lambda$;

(iii) $\underline{D}^*$ is GD with the same parameters as $\underline{D}$;

(iv) $\underline{D}^*$ is GD.

Proof (i) ⇒ (ii) Immediate from Result 1.5.2.

(ii) ⇒ (iii) Suppose $\underline{B}_1,\dots,\underline{B}_d$ is a CLP Division of $\underline{D}^*$ with $\rho=\lambda$.

Summing the identity of Lemma 2.1.6(ii) over all blocks of $\underline{D}$ we obtain :-

$$\sum_{i=1}^{d}\sum_{\substack{j=1\\j\neq i}}^{d} m_i m_j \rho_{ij} = vk(k-1)+v\lambda-\lambda\sum_{i=1}^{d} m_i^2$$

$$= v\ell\lambda+v^2\lambda'-v\ell\lambda'-\lambda\sum_{i=1}^{d} m_i^2 \text{ (by Result 1.4.1(ii)).}$$

Also, by Lemma 2.1.6(iii), we have

$$\sum_{i=1}^{d}\sum_{\substack{j=1\\j\neq i}}^{d} m_i m_j \rho_{ij}^2 = v^2\lambda'^2-v\ell\lambda'^2+v\ell\lambda^2-\lambda^2\sum_{i=1}^{d} m_i^2 .$$

Hence $$\sum_{i=1}^{d}\sum_{\substack{j=1\\j\neq i}}^{d} m_i m_j (\lambda'-\rho_{ij})^2 = (v^2\lambda'^2-\lambda'^2\sum_{i=1}^{d} m_i^2)-2\lambda'(v\ell\lambda+v^2\lambda'$$

$$-v\ell\lambda'-\lambda\sum_{i=1}^{d} m_i^2) + (v^2\lambda'^2-v\ell\lambda'^2+v\ell\lambda^2-\lambda^2\sum_{i=1}^{d} m_i^2)$$

$$= (v\ell-\sum_{i=1}^{d} m_i^2)(\lambda-\lambda')^2 .$$

L.H.S. ≥ 0 and so $v\ell \geq \sum_{i=1}^{d} m_i^2$.

But $0 \leq \sum_{i=1}^{d}(m_i-\ell)^2 = \sum_{i=1}^{d} m_i^2 - v\ell$, and hence $v\ell \leq \sum_{i=1}^{d} m_i^2$.

So $v\ell = \sum_{i=1}^{d} m_i^2$, and thus $m_i = \ell$ for every i.

This ensures $\rho_{ij} = \lambda'$ for every i,j ($i \neq j$) and (iii) follows.

(iii) $\Rightarrow$ (iv) Immediate.

(iv) $\Rightarrow$ (i) Immediate by Theorem 3.2.6. ¤

Remark (iii) $\Rightarrow$ (i) of this theorem is due to Bose and Shrikhande, [15], for the case $|\lambda-\lambda'| = 1$ and to Bose, [13], for general symmetric GD designs.

Theorem 3.3.2 If $\underline{D}$ is symmetric with two intersection numbers : ρ,ρ' ($\rho>\rho'$ if and only if $\lambda>\lambda'$); then the following are equivalent :-

(i) $\underline{D}^*$ is GD with the same parameters as $\underline{D}$;

(ii) $(v-\ell)\rho' + (\ell-1)\rho = k(k-1)$;

(iii) $\rho = \lambda$;

(iv) $\rho' = \lambda'$.

Proof (i) $\Rightarrow$ (ii) Immediate from Result 1.4.1 (ii).

(ii) $\Rightarrow$ (iii) By Result 1.4.1 (ii) : $(v-\ell)\lambda' + (\ell-1)\lambda = k(k-1)$.

Hence $(v-\ell)(\rho'-\lambda') + (\ell-1)(\rho-\lambda) = 0$.

Suppose $\lambda<\lambda'$ and hence $\rho<\rho'$. By Results 3.1.2 and 3.1.3 $\lambda\leq\rho<\rho'$ and by Corollary 3.1.20 either $\lambda=\rho$ and $\lambda'=\rho'$ or $\rho<\lambda'<\rho'$. So if $\lambda\neq\rho$, $\rho-\lambda$ and $\rho'-\lambda'$ are both positive, which cannot occur since $1<\ell<v$. Hence $\rho=\lambda$.

A similar argument holds if $\lambda>\lambda'$, and so we have (iii).

(iii) $\Rightarrow$ (iv) Immediate from Theorem 3.1.17.

(iv) $\Rightarrow$ (i) Immediate from Theorems 3.2.2(ii) and 3.2.5(ii). ¤

We next obtain two well known results using some of the given theory :-

Result 3.3.3 (Shrikhande and Bhagwandas, [45]) If $\underline{D}$ is symmetric and $|\lambda-\lambda'| = 1$, then $\underline{D}^*$ is GD with the same parameters as $\underline{D}$.

Proof Immediate from Corollary 3.1.18 and Theorem 3.3.2. ¤

Remark In fact Shrikhande and Bhagwandas give the stronger result that if $\underline{D}$ is a symmetric PBD(2) with $|\lambda_2-\lambda_1| = 1$, then $\underline{D}^*$ is a PBD(2) with the same parameters as $\underline{D}$.

Result 3.3.4 (Connor, [18]) If $\underline{D}$ is symmetric and regular with $(r^2-v\lambda', \lambda-\lambda') = 1$, then $\underline{D}^*$ is GD with the same parameters as $\underline{D}$.

Proof Immediate from Corollary 3.1.19 and Theorem 3.3.2. ¤

Finally we give two new results giving necessary and sufficient conditions for certain symmetric GD designs to have a GD dual.

Theorem 3.3.5 If $\underline{D}$ is symmetric then either :-

(i) $\underline{D}^*$ is GD with the same parameters as $\underline{D}$; or

(ii) There exist blocks x, x', y, y' satisfying

$$|x\cap x'| < \lambda' < |y\cap y'| \quad \text{and} \quad |x\cap x'| \neq \lambda \neq |y\cap y'|.$$

Proof Immediate from Corollary 3.1.20 and Theorem 3.3.2. ¤

<u>Theorem 3.3.6</u> If $\underline{D}$ is symmetric and either

(i) $\lambda<\lambda'$ and $d=2$, or

(ii) $\lambda>\lambda'=1$,

then $\underline{D}^*$ is GD with the same parameters as $\underline{D}$.

<u>Proof</u> Immediate from Corollary 3.1.21 and Theorem 3.3.2. ¤